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Quasi-exactly solvable 1D quantum problems: location of analytically treatable eigenstates

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1. Introduction

Up to now solvability problem in quantum mechanics is still *terra incognita* neglecting the fact of thousand papers devoted to the topic. Even for the one dimensional case there is no clear understanding of the origin of solubility.

One interesting subject in this field is the case of quasi-exactly solvable quantum problems when only some part of the significant spectrum can be constructed in an analytical way [1].

Quasi-exactly solvable problems play a role of some type of a border dividing region of complete solubility and problems needed numerical computation of their bound states. Among the questions arising here [2, 3], the approach proposed in [4, 5] clarified the fact that for 1D case the quasi-exact solubility leads to a definite polynomial anzatz for a wave functions in such a way that algebraically treated states up to some function providing a proper behaviour at infinity all are polynomial (in respect to a single variable) of the same order with alternating number of real roots starting from zero.

As it was demonstrates in [7], for 1D case we construct a set of eigenstates which always includes the ground state. The origin of such a behaviour is not clear yet, the explicit examples have been investigate in [7]. In the paper we try to add one more example to understand the things better.

We formulate the main goal of the activity as

"to construct a quasi-exactly solvable problem with a set of known eigenstates that will not include the ground state of the system"

We will use the approach developed in [5] as a basis for consideration.

2. Quasi-exactly solvable one dimensional quantum problems

The general form of the second order linear differential equations (SODE) allowing polynomial solutions reads [5]

$$A_{k+2}(x)y''(x) + A_{k+1}(x)y'(x) + A_k(x)y(x) = 0 \tag{1}$$

For $k = 0$ we have the standard eigenvalue (Sturm-Liouville) problem [6]. For $k > 0$, we come to the case of quasi-exactly-solvable problems.

To transform the equation (1) into the Schrödinger equation form we have to implement a pair of transformations [5].

First, this is a variable change

$$\begin{aligned} x &= F(u), \\ \frac{d}{dx} &= \frac{1}{F'(u)}\frac{d}{du}, \\ \frac{d^2}{dx^2} &= \frac{1}{F'^2(u)}\frac{d^2}{du^2} - \frac{F''(u)}{F'^3(u)}\frac{d}{du}, \end{aligned} \tag{2}$$

subjected to the condition

$$[F'(u)]^2 = A_n(x) \tag{3}$$

that leads to

$$y''(u) + A_{n-2}(F(u))y(u) + \frac{y'(u)\left(2\,A_{n-1}(F(u)) + 2\,\sqrt{A_n(F(u))} - A_n'(F(u))\right)}{2\,\sqrt{A_n(F(u))}} = 0 \tag{4}$$

Now we perform a similarity transformation $Y(u) = \exp(\chi(u))y(u)$ and choose $\chi(u)$ in a way to kill the first derivative term. This implies the condition

$$\chi'(u) = \frac{1}{2}\left(\frac{A_n'(F(u))}{2\,\sqrt{A_n(F(u))}} - \frac{A_{n-1}(F(u))}{\sqrt{A_n(F(u))}}\right) \tag{5}$$

and defines the gauge function χ up to a constant.

Then we arrive to the Schrödinger type equation of the form

$$Y''(u) + Y(u)\left[A_{n-2}(F(u)) - \frac{A_{n-1}(F(u))^2}{4\,A_n(F(u))} - \frac{1}{2}A_{n-1}'(F(u)) + \frac{A_{n-1}(F(u))\,A_n'(F(u))}{2\,A_n(F(u))} - \frac{3\,A_n'(F(u))^2}{16\,A_n(F(u))} + \frac{A_n''(F(u))}{4}\right] = 0$$

Let us use this procedure for the case of third order polynomial coefficient functions.

3. Third order polynomial coefficient functions

Let us shortly outline first the case considered in detail in [7]. We choose $A_3(x) = x^3$ that leads to potential expressed in elementary functions rather than general case expressed in elliptical functions. Then from (3) we has $F(u) = 1/u^2$. Let us write down rest coefficient functions as $A_2(x) = \alpha(x-1)(x+1)$ that can be done by appropriate shift and scale of the independent variable x and for $A_1(x) = \gamma + \beta * x$.

Similarity transformation reads

$$\chi(u) = \frac{\alpha u^4}{64} - \alpha\log(u) + \frac{3\log(u)}{2} \tag{6}$$

and Schrödinger equation has the form

$$Y'' + Y\left(\gamma + \frac{-\left(u^4-16\right)^2\alpha^2 + \left(512 - 96u^4\right)\alpha + 64\,(16\beta - 3)}{256u^2}\right) = 0 \tag{7}$$

We search for a polynomial solution of the (1) in the form of the first order polynomials $y(x) = c_1 x + c_0$. Substituting the explicit form of the coefficient function we find the following relation $\beta = -\alpha$ and choosing normalization condition for the eigenfunction as $c_1 = 1$ we get two possible eigenvalues for γ and corresponding solution for c_0

$$\gamma_1 = \alpha \quad c_0^2 = 1 \; \gamma_2 = -\alpha \quad c_0^1 = -1 \tag{8}$$

Then, two eigenfunctions of the Schrödinger equation read

$$Y^{(1)}(u) = \exp\left\{\frac{\alpha u^4}{64}\right\}\left(\frac{4}{u^2}+1\right)u^{\frac{3-2\alpha}{2}}, \tag{9}$$

$$Y^{(2)}(u) = \exp\left\{\frac{\alpha u^4}{64}\right\}\left(\frac{4}{u^2}-1\right)u^{\frac{3-2\alpha}{2}}, \tag{10}$$

To make wave function square integrable and having no singularities at $u = 0$ it is necessary to define admissible range for the parameter α as $\alpha \in]-\infty, -1/2]$. And as one can see, in fact we have two eigenfunctions one corresponding to the ground state (no zeros on the semiaxis $u > 0$) another one corresponding to the first excited state (one zero in the region). Careful analysis manifests that ± 1 in term $u^{-2} \pm 1$ is just the roots (± 1) of the coefficient function $A_2(x)$. As we introduce the coefficient functions with real coefficients we automatically obtain similar behaviour for non-unit value of roots whereas for imaginary roots we can not introduce proper Schrödinger equation as eigenvalues (γ) turned out to be imaginary. The similar results are for search of the arbitrary order polynomials, at the stage of construction of the wave function we have to adjust coefficients values to satisfy standard finite norm condition, energy eigenvalue is inside potential etc. And the adjustment is such that there ground state appears inevitably.

Now, let us consider another case for third order polynomial functions, namely a degenerate cases allowing consideration within the elementary function potentials.

Let $A_3(x) = 1 - x^2$, $A_2(x) = \alpha(1 - x^2)$, $A_1(x) = \gamma + \beta x$.

Then transformation (3) reads $F(u) = \sin u$.

Similarity transformation reads

$$\chi(u) = \frac{1}{8}\sec(u)\left(-4\alpha\sin^2(u) - 4\sin(u) + \alpha\right) \tag{11}$$

and quantum potential for the Schrödinger equation has the form

$$V(u) = \frac{1}{64}\sec^2 u\left(2\cos 4u\alpha^2 + 3\alpha^2 + 48\sin u\alpha - 4\left(\alpha^2+2\right)\cos 2u - \right.$$
$$\left. 16\beta\sin u - 16\beta\sin 3u + 40\right) \tag{12}$$

The first order polynomials solution of (1) is written as $y(x) = c_1 x + c_0$. That again gives $\beta = -\alpha$ and choosing normalization condition for the eigenfunction as $c_1 = 1$ we get two possible eigenvalues for γ and corresponding solution for c_0

$$\gamma_1 = \alpha \quad c_0^2 = 1 \; \gamma_2 = -\alpha \quad c_0^1 = -1 \tag{13}$$

Then, two eigenfunctions of the Schrödinger equation read

$$Y^{(1)}(u) = e^{\frac{1}{2}\alpha\sin(u)}\sqrt{\cos(u)}(\sin(u) - 1) \tag{14}$$

$$Y^{(2)}(u) = e^{\frac{1}{2}\alpha\sin(u)}\sqrt{\cos(u)}(\sin(u) + 1) \tag{15}$$

From the potential behaviour we conclude that the system should be considered in the interval $u \in [-\pi/2, \pi/2]$.

The fast look at (14) proves that the parameter α is not important as it is not responsible for significant singulatities of $V(u)$.

It is also clear from (**??**) that both functions have no nodes in the interval. The only way to expel one is to check that for Y^2 the appropriate eigenvalue $\gamma = -1$ whereas minimum of the potential lays higher. Then again we reject Y^2 and obtain the ground state included into found spectrum.

Though there were additional possibilities we missed, namely we could incorporate second scale into x like term $\alpha(x - x_0)(x + x_0)$ but this will be a topic of separate publication.

4. Conclusion

On explicit examples it was demonstrated that the set of eigenstates for quasi-exactly solvable one dimensional quantum problem always includes the ground state of the system.

References

[1] Bender C M and Dunne G V , Quasi-Exactly Solvable Systems and Orthogonal Polynomials, *J. Math. Phys.* **37** (1996) 6

[2] Gangopadhyaya A and Khare A , Methods for Generating Quasi-Exactly Solvable Potentials, *Phys.Lett.* **A208** (1995) 261-268

[3] Khare A and Mandal B P, Do quasi-exactly solvable systems always correspond to orthogonal polynomials? *Phys. Lett.* **A239** (1998) 197-200

[4] G. Krylov, M. Robnik On 1D Schrodinger problems allowing polynomial solutions *J. Phys.* , **A33** (2000) 1233-1245

[5] G. Krylov, M. Robnik Polynomial families and Schrodinger equation: one example of nonhypergeometric type of correspondence *J. Phys.* , **A34**, (2001) 5403-5415

[6] Morse P M and Feshbach H 1953 *Methods of theoretical physics*, **Vol. 1,2** (New York: McGraw Publ. Company).

[7] George Krylov *On one puzzle in the theory of quasi-exactly solvable problems in quantum mechanics* // in Porceedings of the 5th International Conference BGL-5, Oct. 10-13, 2006, Minsk, Belarus: eds. Yu. Kurochkin and V.M. Red'kov, IP NAS Minsk, 2006, p.286-289

[8] Turbiner A Lie algebras and polynomials in one variable *J. Phys. A: Math. Gen.* **25** (1992) L1087-L1093