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A projection operators technique in calculations of Green function for quantum system in external field

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Properties of projective operational function of Green for quantum system was analysed. The diagrammatic technique to find a Green function of quantum system in an external field is offered.

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1. Introduction

The method of Green functions is one of the most powerful methods of computing physics. In [1] we have shown that a Green function of quantum system is represented by matrix elements of the operator function expressing through projection operators. We shall call this operator function as a projection operator Green function and further we shall examine its properties.

The goal of the article is to develop a diagram technique which allows to find the projection operator Green function describing a quantum system in an external field.

2. Formalism of Green function

Green function $G(\vec{r},\vec{r}';t,t')$ satisfies to an equation with in a δ-function in the right part, for example, as

$$\left[\imath\hbar\frac{\partial}{\partial t}-\hat{H}(\vec{r},\vec{r}_1,\vec{r}_2,\ldots;t)\right]G(\vec{r},\vec{r}';t,t')=\delta(\vec{r}-\vec{r}')\delta(t-t'), \tag{1}$$

where $\vec{r},\vec{r}_i,\vec{r}'$ is space variables, t,t' is time. Since $G(\vec{r},\vec{r}';t,t')$ should describe propagation of a signal from a point $\vec{r}'$ to a point $\vec{r}$ for a time interval $(t-t')$, it depends on a difference of times $(t-t')$:

$$G(\vec{r},\vec{r}';t,t')\equiv G(\vec{r},\vec{r}';t-t'). \tag{2}$$

The time interval $\tau=(t-t')$ should be positive as owing to the wave nature the signal is propagated according to a Huygens principle, i.e. an excitation is transferred sequentially in time from one point of space to the nearest one. Therefore, the Green function has no physical sense itself. Physical sense has a propagation function or propagator $G^{\pm}(\vec{r},\vec{r}';\tau)$ determined as

$$G^{\pm}(\vec{r},\vec{r}';\tau)=\theta(\pm\tau)G(\vec{r},\vec{r}';\tau), \tag{3}$$

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where the function $G^{+}(\vec{r},\vec{r}';\tau)$ is called a retarded Green function, the function $G^{-}(\vec{r},\vec{r}';\tau)$ is called a advanced Green function. A Laplace transformation of Green function $G(\vec{r},\vec{r}';t-t')$ has the following form

$$G(\vec{r},\vec{r}';t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-\imath\omega(t-t')} G(\vec{r},\vec{r}';\omega). \tag{4}$$

Let us examine a case of a hamiltonian $\hat{H} \equiv \hat{H}_0(\vec{r},\vec{r}_1,\vec{r}_2,\ldots)$ which is independent of time t. Then the Laplace image of function of Green $G(\vec{r},\vec{r}';z)$, being analytical continued in a complex plane Z, can be determined as a solution of the following non-homogeneous differential equation:

$$[z - \hat{H}_0(\vec{r},\vec{r}_1,\vec{r}_2,\ldots)]G(\vec{r},\vec{r}';z) = \delta(\vec{r}-\vec{r}') \tag{5}$$

which has boundary conditions defined on boundary surface S of area V. Here we assume that z is a complex variable, the time independent operator $\hat{H}_0(\vec{r})$ is a linear Hermitian differential operator which possesses a complete set of eigen functions $\{\phi_n(\vec{r})\}$:

$$\hat{H}_0(\vec{r},\vec{r}_1,\vec{r}_2,\ldots)\phi_n(\vec{r}) = \omega_n\phi_n(\vec{r}). \tag{6}$$

The set $\{\phi_n(\vec{r})\}$ can be orthonormalized

$$\int \phi_n(\vec{r})^{\dagger}\phi_m(\vec{r})d\vec{r} = \delta_{nm}, \tag{7}$$

where "†" denotes Hermitian conjugation. Completeness of the set $\{\phi_n(\vec{r})\}$ means that

$$\sum_n \phi_n(\vec{r})\phi_n^{\dagger}(\vec{r}') = \delta(\vec{r}-\vec{r}'). \tag{8}$$

The symbol $\sum_n$ it is necessary to interpret as $\sum_n' + \int dn$ if index n numbers the eigen functions belonging both discrete, and a continuous spectrum.

It is convenient to work in abstract vector space. This representation is named a representation of bra (ket)- Dirac vectors [2], [3], [4], [5]. Using designations of the Dirac bra (ket) - vectors ($\langle\vec{r}|$, $\langle\phi_n|$, $|\vec{r}\rangle$, $|\phi_n\rangle$), in such space one can constrict operators of projection (projectors) P

$$|\vec{r}\rangle\langle\vec{r}|,\ |\phi_n\rangle\langle\phi_n|. \tag{9}$$

Then, for example, $\phi_n(\vec{r})$ is a projection of the vector $|\phi_n\rangle$ onto the vector $|\vec{r}\rangle$:

$$|\phi_n\rangle = \int d\vec{r}|\vec{r}\rangle\langle\vec{r}|\phi_n\rangle = \int d\vec{r}\phi_n(\vec{r})|\vec{r}\rangle. \tag{10}$$

The equality (10) gives two following identities

$$\int d\vec{r}|\vec{r}\rangle\langle\vec{r}| = \hat{1}, \tag{11}$$

$$\sum_n |\phi_n\rangle\langle\phi_n| = \hat{1}, \tag{12}$$

where $\hat{I}$ is unity operator. From the property of completeness (8) and the expression (12) the following proof of orthonormality for the basis $\{|\vec{r}\rangle\}$ is ensued

$$\langle\vec{r}'|\vec{r}\rangle = \sum_n \langle\vec{r}'|\phi_n\rangle\langle\phi_n|\vec{r}\rangle \sum_n \phi_n(\vec{r})\phi_n^{\dagger}(\vec{r}') = \delta(\vec{r}-\vec{r}'). \tag{13}$$

It is easy to prove that the operator $\hat{H}_0$ in this representation takes the matrix form $\langle\vec{r}'|\hat{H}_0|\vec{r}\rangle$:

$$\hat{H}_0 = \hat{I}^2\hat{H}_0 = \int\int d\vec{r}d\vec{r}'|\vec{r}'\rangle\langle\vec{r}'|\hat{H}_0|\vec{r}\rangle\langle\vec{r}|, \tag{14}$$

and the Green function $G(\vec{r},\vec{r}';z)$ can be considered as a matrix element of the operator $\hat{G}(z)$:

$$\begin{aligned}\hat{G}(z) = \hat{I}^2\hat{G}(z) &= \int\int d\vec{r}d\vec{r}'|\vec{r}'\rangle\langle\vec{r}'|\hat{G}(z)|\vec{r}\rangle\langle\vec{r}| \\ &\equiv \int\int d\vec{r}d\vec{r}'|\vec{r}'\rangle G(\vec{r},\vec{r}';z)\langle\vec{r}|.\end{aligned} \tag{15}$$

One can conclude from the expressions (14) and (15) that $\hat{G}(z), \hat{H}_0$ are projectors in this abstract space.

According to (14), (15) and (13) the equation (5) can be rewritten in the form

$$\begin{aligned}[z - H_0(\vec{r},\vec{r}_1,\vec{r}_2,\ldots)]G(\vec{r},\vec{r}';z) &= [z - (H_0(\vec{r}',\vec{r}_1,\vec{r}_2,\ldots)\delta(\vec{r}-\vec{r}'))]G(\vec{r},\vec{r}';z) \\ &= zG(\vec{r},\vec{r}';z) - \langle\vec{r}'|[\hat{H}_0G(z)](\vec{r},\vec{r}')|\vec{r}\rangle = \langle\vec{r}'|z\hat{G}(z)|\vec{r}\rangle \\ &- \int d\vec{r}''\langle\vec{r}'|\hat{H}_0|\vec{r}''\rangle\langle\vec{r}''|\hat{G}(z)|\vec{r}\rangle = \langle\vec{r}'|\vec{r}\rangle\end{aligned} \tag{16}$$

Taking into account the expression (11), the equation (16) is rewritten in the operator form

$$[z - \hat{H}_0]\hat{G}(z) = 1. \tag{17}$$

If all eigenstate of the operator are non-zero, the equation (17) can be solved formally as

$$\hat{G}(z) = \frac{1}{z - \hat{H}_0}. \tag{18}$$

Multiplying (18) on the expression (12), we get

$$\hat{G}(z) = \sum_n \frac{1}{z - \hat{H}_0}|\phi_n\rangle\langle\phi_n| = \sum_n \frac{|\phi_n\rangle\langle\phi_n|}{z - \omega_n}. \tag{19}$$

Hence $\hat{H}$ is the hermitre operator, all its eigenvalues ω_n are real. Therefore of $\Im m\ z \neq 0$, then $z \neq \omega_n$. The last means that $G(z)$ is an analytical function in complex Z-plane everywhere except points sand intervals on real axis that correspond to eigenvalues of $\hat{H}$.

Let us define $z = \omega \pm \imath\epsilon$. Then the expression (19) gives two real functions

$$G^+(\vec{r},\vec{r}';\omega) = \lim_{\epsilon\to 0} G(\vec{r},\vec{r}';\omega + \imath\epsilon) = \sum_n \frac{\phi_n^*(\vec{r})\phi_n(\vec{r}')}{\omega - (\omega_n - \imath\epsilon)}, \tag{20}$$

$$G^-(\vec{r},\vec{r}';\omega) = \lim_{\epsilon\to 0} G(\vec{r},\vec{r}';\omega - \imath\epsilon) = \sum_n \frac{\phi_n^*(\vec{r})\phi_n(\vec{r}')}{\omega - (\omega_n + \imath\epsilon)}. \tag{21}$$

Let us clarify the physical meaning of these functions. Let us finc the integral

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-\imath\omega t}G^+(\vec{r},\vec{r}';\omega + \imath\epsilon) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-\imath\omega t}\sum_n \frac{\phi_n^*(\vec{r})\phi_n(\vec{r}')}{\omega - (\omega_n - \imath\epsilon)}. \tag{22}$$

If one assumes $t<0$ and choses a countour C_1 with counterclockwise direction that is close it in upper semi-plane (Fig. 3a) then there are no poles inside C_1 and therefore the integral (23) is equal to zero

$$\int_{C_1}\frac{dz}{2\pi}e^{\imath z|t|}\sum_n\frac{\phi_n^*(\vec{r})\phi_n(\vec{r}')}{z-(\omega_n-\imath\epsilon)}. \tag{23}$$

If one assumes $t>0$, then integrand in (23) should be replaced on

$$e^{\imath(-z)|t|}\sum_n\frac{\phi_n^*(\vec{r})\phi_n(\vec{r}')}{z-(\omega_n-\imath\epsilon)}. \tag{24}$$

Then, according to (24) phase of the integration variable $z=\rho\exp(\imath\phi)$ is changed on π at change $t\to -t$, therefore for positive time interval we have the integral

$$\begin{aligned}&\int_{-C_2}\frac{1}{2\pi}d\rho d[\exp(\imath(\phi+\pi))]e^{\imath(-z)|t|}\sum_n\frac{\phi_n^*(\vec{r})\phi_n(\vec{r}')}{z-(\omega_n-\imath\epsilon)}\\ &=-\int_{-C_2}\frac{1}{2\pi}d\rho d[\exp(\imath\phi)]e^{\imath(-z)|t|}\sum_n\frac{\phi_n^*(\vec{r})\phi_n(\vec{r}')}{z-(\omega_n-\imath\epsilon)}\\ &=\int_{C_2}\frac{dz}{2\pi}e^{-\imath zt}\sum_n\frac{\phi_n^*(\vec{r})\phi_n(\vec{r}')}{z-(\omega_n-\imath\epsilon)},\end{aligned} \tag{25}$$

where C_2 is arbitrary contour with the clockwise direction. From that it follows that in comparison with the case $t<0$ contour is closed in lower semi-plane is shown in in Fig.3b. Therefore the poles are inside the contour and the integral (25) does not vanish. Thus we finally get

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-\imath\omega t}G^+(\vec{r},\vec{r}';\omega+\imath\epsilon)=\theta(t)\int_{-C}\frac{dz}{2\pi}e^{-\imath zt}G(\vec{r},\vec{r}';z), \tag{26}$$

where C is an arbitrary contour with counterclockwise direction included poles of integrand in the expression ().

In a similar way, using Fig.4, one can find the integral

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-\imath\omega t}G^-(\vec{r},\vec{r}';\omega-\imath\epsilon)=\theta(-t)\int_{C}\frac{dz}{2\pi}e^{-\imath zt}G(\vec{r},\vec{r}';z), \tag{27}$$

where C is an arbitrary contour with counterclockwise direction included poles of integrand in the expression (27) as shown in in Fig. 4b. Comparing (26, 27) with (3), we conclude that $G^+(\vec{r},\vec{r}';\omega+\imath\epsilon)$ is the Laplace image of the retarded, and $G^-(\vec{r},\vec{r}';\omega-\imath\epsilon)$ is the Laplace image of the advanced Green functions.

Let us place one part of the poles of the function $G(z)$ using shifts on $+\imath\epsilon$, whereas the second part by shifts on $-\imath\epsilon$, as it is shown in Fig 5a. Then the poles of $G(z,\lambda_1-\imath\epsilon,\lambda_2-\imath\epsilon,\lambda_3+\imath\epsilon,\dots,\lambda_n+\imath\epsilon,)$ are inside integration contour for all t. Function $G(z,\lambda_1-\imath\epsilon,\lambda_2-\imath\epsilon,\lambda_3+\imath\epsilon,\dots,\lambda_n+\imath\epsilon,)$ is called the Laplace image of the causal Green function.

As one can see from Fig. 5 the difference of contours C_1,C_2 can be deformed to contour that include the real axis $\Re eZ$. Therefore, at integration on C_1-C_2 one can omit shift of poles on $\pm\epsilon$. Therefore the causal Green function $G(\vec{r},\vec{r}';t)$ can be written as

$$G(\vec{r},\vec{r}';t)=-\int_{C_2-C_1}\frac{dz}{2\pi}e^{-\imath zt}G(\vec{r},\vec{r}';z). \tag{28}$$

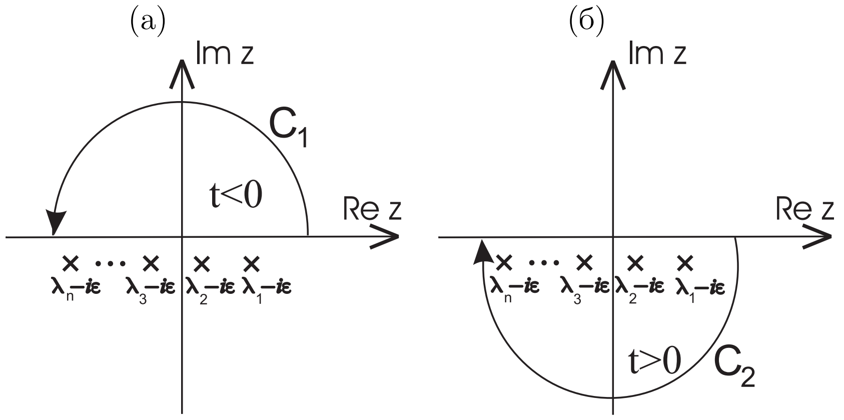


Fig. 3. Contour of integration for retarded Green function. (a) Integration contour for negative time $t,\ t<0$. (b) (a) Integration contour for positive time

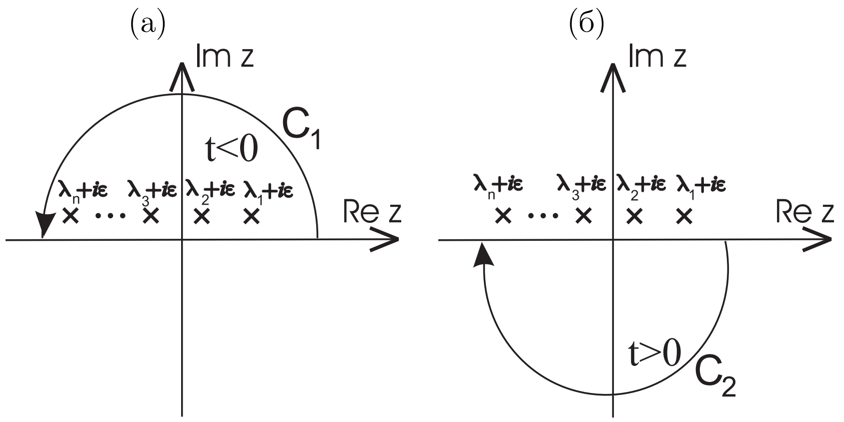


Fig. 4. Integration contour for advanced Green function. (a) Integration contour for negative time $t,\ t<0$. (b) (a) Integration contour for positive time

If we deform contours C_1, C_2 in Fig. 5a in such a way that the resulting contour C_1' includes poles $\lambda_3+\imath\epsilon, \ldots, \lambda_n+\imath\epsilon$, and contour C_2', obtained from C_2, includes the poles λ_1, λ_2, then the causalfunction $G(\vec{r}, \vec{r}\,'; t)$ can be expressed through functions $G^{\pm}$:

$$G(\vec{r}, \vec{r}\,'; t) = -\theta(-t)\int_{C_1'} \frac{dz}{2\pi} e^{-\imath zt} G(\vec{r}, \vec{r}\,'; z-\imath\epsilon)$$
$$-\theta(t)\int_{-C_2'} \frac{dz}{2\pi} e^{-\imath zt} G(\vec{r}, \vec{r}\,'; z+\imath\epsilon) = G^{+}(\vec{r}, \vec{r}\,'; t) - G^{-}(\vec{r}, \vec{r}\,'; t). \tag{29}$$

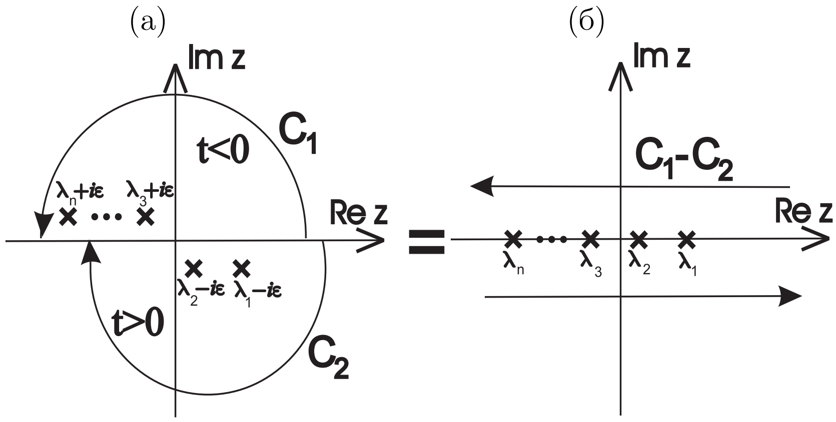
3. Evolution operator

Let us demonstrate that the expressions (28) and (29) are equivalent. With this goal invmind we use the Sohotsky identity

$$\lim_{\epsilon\to 0} \frac{1}{x \pm \imath\epsilon} = P\frac{1}{x} \mp \imath\pi\delta(x), \tag{30}$$

to find the difference of Laplace images of retarded and advanced Green functions. The the Laplace images $G^{\pm}(\vec{r},\vec{r}\,';\omega)$ have the form

$$G^{\pm}(\vec{r},\vec{r}\,';\omega) = P\sum_n \frac{\phi_n^*(\vec{r})\phi_n(\vec{r}\,')}{\omega-\omega_n} \mp \imath\pi\sum_n \phi_n^*(\vec{r})\phi_n(\vec{r}\,')\delta(\omega-\omega_n) \tag{31}$$



үис. 5. үонтур интегрирования для причинной функции +рина. (а) үонтура интегрирования для отрицательнvх времен $t,\ t<0$ и положительнvх времен $t,\ t>0$. (б) Contour of integration in the case of poles position at real axis. Substituting (31) into (29), we get

$$\begin{aligned} G(\vec{r},\vec{r}\,';t) &= G^{+}(\vec{r},\vec{r}\,';t) - G^{-}(\vec{r},\vec{r}\,';t) \\ = -2\pi\imath\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-\imath\omega t}\sum_n \delta(\omega-\omega_n)\phi_n^*(\vec{r})\phi_n(\vec{r}\,') &= -\imath\sum_n e^{-\imath\omega_n t}\phi_n^*(\vec{r})\phi_n(\vec{r}\,'). \end{aligned} \tag{32}$$

Using the expressions (12), (19) and (32), one can easily show that the function $G(\vec{r},\vec{r}\,';t)$ is a matrix element of the operator $\hat{G}(t)$, defined by the expression

$$\hat{G}(t) = -\imath\sum_n e^{-\imath\omega_n t}|\phi_n\rangle\langle\phi_n| = -\imath e^{-\imath\hat{H}_0 t}. \tag{33}$$

Let us put $t = t_2 - t_1$. The the matrix form of the expression (33) gives the expression for the Green function

$$\begin{aligned} G(\vec{r},\vec{r}\,';t_2,t_1) &= -\imath\sum_n \langle\vec{r}|e^{-\imath\omega_n t_2}|\phi_n\rangle\langle e^{\imath\omega_n t_1}\phi_n|\vec{r}\,'\rangle \\ &= -\imath\sum_n \phi_n(\vec{r},t_2)\phi_n^*(\vec{r}\,',t_1). \end{aligned} \tag{34}$$

Here in the right hand side of the expression (28) we change matrix element $G(\vec{r},\vec{r}\,';z)$ of projective operator $\hat{G}(z)$ on operator itself $\hat{G}(z)$, given by the expression (18). Then using the residue theorem at formal integration we obtain the projective operator

$$\hat{G}_0(t) = -\imath\int_{C_2-C_1}\frac{dz}{2\pi\imath}e^{-\imath zt}\frac{1}{z-\hat{H}_0} = -\imath e^{-\imath t\hat{H}_0}, \tag{35}$$

which coincides with (33). Therefore one concludes that the expressions (28) and (29) are equivalent. But the advantage of the first expression (28) for the causal green function on respect to the second one (29) for the same function is in symmetric entrance of space and time coordinate into the first expression.

Let us shown that the projective operator

$$\hat{U}(t-t_0) = e^{-\imath(t-t_0)\hat{H}_0} = \imath\hat{G}(t-t_0) \tag{36}$$

is the projective operator of evolution. Indeed in the representation of bra- ket Dirac vectors the function

$$|\phi(t)\rangle = \hat{U}(t-t')|\phi(t')\rangle = e^{-\imath(t-t')\hat{H}}|\phi(t')\rangle \tag{37}$$

satisfies the equation

$$\imath\frac{\partial}{\partial t}\langle\vec{r}|\phi(t)\rangle - \int d\vec{r}\,'\langle\vec{r}|\hat{H}_0\vec{r}\,'\rangle\langle\vec{r}\,'|\phi(t')\rangle = 0, \quad \hbar = 1. \tag{38}$$

According to definition (14) the equation (38) describes the systems behaviour in coordinate representation

$$\left[\imath\frac{\partial}{\partial t} - \hat{H}_0(\vec{r})\right]\phi(\vec{r},t) = 0, \quad \hbar = 1 \tag{39}$$

But the equation (39) describes the evolution of a wave function in time. therefore the projective operator of evolution $\hat{U}(t-t_0)$ propagates $|\phi\rangle$ from the time moment t_0 to the time moment t.

Hence the projector $\hat{U}$ is the operator exponent it possesses important property

$$\hat{U}(t-t_0) = \hat{U}(t-t_1)\hat{U}(t_1-t_0), \tag{40}$$

which allows to introduce the diagram technique in the following paragraph.

4. Diagram expansion for the Green function in representation of projective operators. The case of an external field.

Let a system be subjected by time dependent external effect.
$\hat{H}_1(\vec{r},t)$, is described by the equation (39).

Then the Hamiltonian of the system can be represented in the form

$$\hat{H}(\vec{r},t) = \hat{H}_0(\vec{r}) + \hat{H}_1(\vec{r},t), \tag{41}$$

where $H_0(\vec{r})$ is a Hamiltonian of the unperturbed system. In representation of interaction the wave function is defined as

$$\phi(\vec{r},t) = e^{-\imath\hat{H}_0 t}\phi^V(\vec{r},t), \tag{42}$$

where $\phi^V(\vec{r})$ is a wave function $\phi(\vec{r})$ in the representation of interaction. Substituting the expressions (41), (42) into equation (39) and multiplying this equation from the left on $\exp(\imath\hat{H}_0 t)$, we get

$$e^{\imath\hat{H}_0 t}\left[\imath\frac{\partial}{\partial t} - \hat{H}_0(\vec{r}) - \hat{H}_1(\vec{r},t)\right]e^{-\imath\hat{H}_0 t}\phi^V(\vec{r},t) = 0, \quad \hbar = 1. \tag{43}$$

From (43) we find that in the representation of interaction the equation представлении взаимодействия уравнение (39) gets the form

$$\left[\imath\frac{\partial}{\partial t} - \hat{H}_1^V(\vec{r},t)\right]\phi^V(\vec{r},t) = 0, \quad \hbar = 1, \tag{44}$$

where $\hat{H}_1^V$ is defined by the expression

$$\hat{H}_1^V = e^{\imath t\hat{H}_0}\hat{H}_1 e^{-\imath t\hat{H}_0}, \quad \hbar = 1. \tag{45}$$

Let perturbation $\hat{H}_1(\vec{r},t)$ be small enough and the Green function$G_1^{(0)}(\vec{r}_2,t_2;\vec{r}_1,t_1)$ of non-interacting system with a Hamiltonian гамильтонианом $\hat{H}_0(\vec{r})$ is known. Let us call this Green function as a one-particle free Green function as the system with one degree of freedom is considered.

Then in the representation of interaction in accord with formulae (36) and (44) the projective operator $\hat{G}_1^V(t_i - t_{i-1})$ for the Green function which describes the signal propagation for the infinitely small time interval Δt_i and defined by the Hamiltonian $\hat{H}(t)$, can be represented in the form of perturbation series expansion as

$$\begin{gathered}\imath\hat{G}_1^V(t_i - t_{i-1}) = e^{-\imath\hat{H}_1^V(\tilde{t}_i)\Delta t_i} = (1 - \imath\hat{H}_1^V(t_i)\Delta t_i), \\ \Delta t_i = t_i - t_{i-1} = \epsilon \to 0,\ \tilde{t}_i \in \{t_{i-1}, t_i\}.\end{gathered} \tag{46}$$

Using the property (40) of the evolution operator we rewrite the equation in the form (46):

$$\begin{aligned}\imath\hat{G}_1^V(t_k - t_{i-1}) &= e^{-\imath(t_k - t_i)\hat{H}_1^V(t_k)}e^{-\imath(t_i - t_{i-1})\hat{H}_1^V(t_i)} = \imath\hat{G}_1^V(t_k - t_i)\ \imath\hat{G}_1^V(t_i - t_{i-1}) \\ &= \Big[1 - \imath(\hat{H}_1^V(t_i)\Delta t_i + \hat{H}_1^V(t_k)\Delta t_k) - \hat{H}_1^V(t_i)\Delta t_i\hat{H}_1^V(t_k)\Delta t_k \\ &\qquad -\hat{H}_1^V(t_k)\Delta t_k\hat{H}_1^V(t_i)\Delta t_i\Big], \quad \Delta t_i = \Delta t_k = \epsilon \to 0.\end{aligned} \tag{47}$$

Performing the above described procedure n times and performing $n \to \infty$, we get the projective Green function which describes the signal propogation starting from time moment t_0 up to t_1:

$$\begin{gathered}\imath\hat{G}_1^V(t_1 - t_0) = e^{-\imath\sum_j(t_j - t_{j-1})\hat{H}_1^V(t_j)} \\ = \left[1 - \imath\int\hat{H}_1^V(t_i)dt_i - \int\int\hat{H}_1^V(t_i)dt_i\hat{H}_1^V(t_k)dt_k - \dots\right].\end{gathered} \tag{48}$$

Now, using the transformation (45) and (42), we can turn back from the representation of interaction into a standard one and find the projective Green function $\hat{G}(t_1 - t_0)$ as:

$$\begin{gathered}\imath\hat{G}_1(t_1 - t_0) = e^{-\imath(t_1 - t_0)\hat{H}_0} - \imath\int e^{-\imath(t_1 - t_i)\hat{H}_0}\hat{H}_1(t_i)e^{-\imath(t_i - t_0)\hat{H}_0}dt_i \\ -\int\int dt_i\ dt_k e^{-\imath(t_1 - t_i)\hat{H}_0}\hat{H}_1(t_i)e^{-\imath(t_i - t_k)\hat{H}_0}\hat{H}_1(t_k)e^{-\imath(t_k - t_0)\hat{H}_0} - \dots.\end{gathered} \tag{49}$$

Remarking that the expression $\{-\imath\exp[-\imath(t_j - t_l)\hat{H}_0]\}$ is the projective Green function of a free particle $\hat{G}_1^{(0)}(t_j - t_l)$, we finally find that

$$\begin{gathered}\hat{G}_1(t_1 - t_0) = \hat{G}_1^{(0)}(t_1 - t_0) + \int\hat{G}_1^{(0)}(t_1 - t_i)\hat{H}_1(t_i)\hat{G}_1^{(0)}(t_i - t_0)dt_i \\ +\int\int dt_i\ dt_k\hat{G}_1^{(0)}(t_1 - t_i)\hat{H}_1(t_i)\hat{G}_1^{(0)}(t_i - t_k)\hat{H}_1(t_k)\hat{G}_1^{(0)}(t_k - t_0) + \dots.\end{gathered} \tag{50}$$

Now we rewrite the expression (50) in matrix form

$$\begin{aligned} G_1(\vec{r}_1,t_1;\vec{r}_0,t_0) &= G_1^{(0)}(\vec{r}_1,t_1;\vec{r}_0,t_0) \\ &+ \int d\vec{r}_i\, dt_i G_1^{(0)}(\vec{r}_1,t_1;\vec{r}_i,t_i)\hat{H}_1(\vec{r}_i,t_i)G_1^{(0)}(\vec{r}_i,t_i;\vec{r}_0,t_0) \\ &+ \int\int d\vec{r}_i\, dt_i\; d\vec{r}_k\, dt_k G_1^{(0)}(\vec{r}_1,t_1;\vec{r}_i,t_i)\hat{H}_1(\vec{r}_i,t_i)G_1^{(0)}(\vec{r}_i,t_i;\vec{r}_k,t_k)\hat{H}_1(\vec{r}_k,t_k) \\ &\times G_1^{(0)}(\vec{r}_k,t_k;\vec{r}_0,t_0) + \ldots . \end{aligned} \tag{51}$$

Now we point out that the space and time variables are included into equation (51) in a symmetric way. Then one can rewrite the equation (51) in a covariant form

$$\hat{G}_1 = \hat{G}_1^{(0)} + \hat{G}_1^{(0)}\hat{H}_1\hat{G}_0 + \hat{G}_1^{(0)}\hat{H}_1\hat{G}_1^{(0)}\hat{H}_1\hat{G}_1^{(0)} + \ldots . \tag{52}$$

where

$$\hat{H}_1 = \hat{I}^2\hat{H}_1 = \int\int dx dx' |x'\rangle\langle x'|\hat{H}_1|x\rangle\langle x|, \tag{53}$$

$x = \{\vec{r}, t\}$, $x' = \{\vec{r}', t'\}$

$$\hat{G}_1 = \hat{I}^2\hat{G}_1 = \int\int dx dx' |x'\rangle\langle x'|\hat{G}_1|x\rangle\langle x| \equiv \int\int dx dx' |x'\rangle G_1(\vec{r},t;\vec{r}\,',t')\langle x|. \tag{54}$$

Let us designate the free green function $\hat{G}_1^{(0)}$ by a solid line, perturbed Green function $\hat{G}_1$ by a double solid line and perturbation $\hat{H}_1$ by a pointed line. Then one can correspond a diagram to the equation (52) as shown in Fig. 6.

In the form represented the perturbation $\hat{H}_1(\vec{r}_i, t_i)$ is an external filed. Beside, the equation (52) is precisely coincided in form with the Dyson-S equation if one rewrite it as

$$\hat{G}_1 = \hat{G}_1^{(0)} + \hat{G}_1^{(0)}\Sigma_1\hat{G}_1^{(0)}, \tag{55}$$

$$\Sigma_1 = \hat{H}_1\hat{G}_1^{(0)} + \hat{G}_1^{(0)}\hat{H}_1\hat{G}_1^{(0)}\hat{H}_1 + \ldots ; \tag{56}$$

where Σ_1 is the self-energy operator. therefore the expansion (55) is the Dyson-Schwinger equation for particles in an external filed.

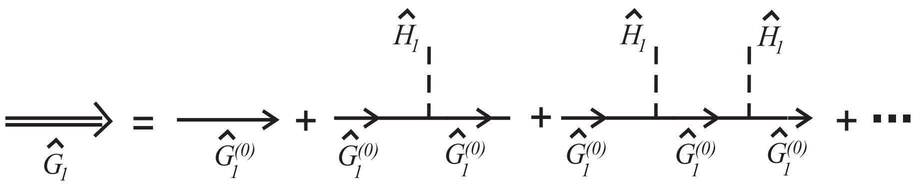


Fig. 6. Diagram for calculation of the one-particle Green function $\hat{G}_1$ up to the second order on external perturbation $\hat{H}_1$.

5. Conclusion

Within the technique of projection operators we have offered the rigorous procedure to construct diagrammatic expansions for operator Green functions. This allows us a propagator of quantum system in external field.

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