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On parametrization of some sub-groups of the unitary group SU(4)

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Parametrization of 4×4 – matrices G of the complex linear group $GL(4.C)$ in terms of four complex vector-parameters $G = G(k, m, n, l)$ is investigated. Additional restrictions separating some sub-groups of $GL(4.C)$ are given explicitly. In the given parametrization, the problem of inverting any 4×4 matrix G is solved. Expression for determinant of any matrix G is found: det $G = F(k, m, n, l)$. Unitarity conditions in the theory of 4×4-matrices on the base of complex vector parametrization in the theory of the group $GL(4.C)$ is investigated. Unitarity conditions have been formulated in the form of non-linear cubic algebraic equations including complex conjugation. Two simplest types of solutions have been constructed: 1-parametric Abelian sub-group G_0 of 4×4 unitary matrices; three 2-parametric sub-groups G_1, G_2, G_3; one 4-parametric unitary sub-group. Curvilinear coordinates to cover these sub-groups have been found.

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1. On parameters of inverse transformations $G \in GL(4.C)$

Arbitrary 4×4 matrix $G \in GL(4.C)$ can be decomposed in terms of 16 Dirac algebra basis:

$$G = A\, I + iB\, \gamma^5 + iA_l\, \gamma^l + B_l\, \gamma^l\gamma^5 + F_{mn}\, \sigma_{mn}\ , \tag{1}$$

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2g^{ab}\ , \qquad \gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3\ ,$$

$$\sigma^{ab} = \frac{1}{4}\,(\gamma^a\gamma^b - \gamma^b\gamma^a)\ , \qquad g^{ab} = \text{diag}(+1, -1, -1, -1)\ . \tag{2}$$

Taking 16 coefficients A, B, A_l, B_l, F_{mn} as independent parameters in the group $GL(4.C)$:

$$G \in GL(4.C): \qquad G = G(A, B, A_l, B_l, F_{mn})\ .$$

one can establish the corresponding multiplication law for these parameters [1, 2] :

$$G' = A'\, I + iB'\, \gamma^5 + iA'_l\, \gamma^l + B'_l\, \gamma^l\gamma^5 + F'_{mn}\, \sigma_{mn}\ ,$$
$$G = A\, I + iB\, \gamma^5 + iA_l\, \gamma^l + B_l\, \gamma^l\gamma^5 + F_{mn}\, \sigma_{mn}\ ,$$
$$G'G = A''\, I + iB''\, \gamma^5 + iA''_l\, \gamma^l + B''_l\, \gamma^l\gamma^5 + F''_{mn}\, \sigma_{mn}\ ,$$
$$<\ (A', B', A'_l, B'_l, F'_{mn}), (A, B, A_l, B_l, F_{mn})\ > = (A'', B'', A''_l, B''_l, F''_{mn})\ ,$$

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where

$$A'' = A'\ A - B'\ B - A'_l\ A^l - B'_l\ B^l - \frac{1}{2}\ F'_{kl}\ F^{kl}\ ,$$

$$B'' = A'\ B + B'\ A + A'_l\ B^l - B'_l\ A^l + \frac{1}{4}\ F'_{mn}\ F_{cd}\ \epsilon^{mncd}\ ,$$

$$A''_l = A'\ A_l - B'\ B_l + A'_l\ A + B'_l\ B + A'^k F_{kl} +$$
$$+F'_{lk} A^k + \frac{1}{2}\ B'_k\ F_{mn}\ \epsilon_l{}^{kmn} + \frac{1}{2}\ F'_{mn}\ B_k\ \epsilon_l{}^{mnk}\ ,$$

$$B''_l = A'\ B_l + B'\ A_l - A'_l\ B + B'_l\ A + B'^k\ F_{kl} +$$
$$+F'_{lk}\ B^k + \frac{1}{2}\ A'_k\ F_{mn}\ \epsilon^{kmn}{}_l + \frac{1}{2}\ F'_{mn}\ A_k\ \epsilon^{mnk}{}_l\ ,$$

$$F''_{mn} = A'\ F_{mn} + F'_{mn}\ A - (A'_m\ A_n - A'_n\ A_m) - (B'_m\ B_n - B'_n\ B_m) +$$
$$+A'_l\ B_k\ \epsilon^{lkmn} - B'_l\ A_k\ \epsilon^{lkmn} + \frac{1}{2}\ B'\ F_{kl}\ \epsilon^{kl}{}_{mn} + \frac{1}{2}\ F'_{kl}\ B\ \epsilon^{kl}{}_{mn} +$$
$$+(F'_{mk}\ F^k{}_n - F'_{nk}\ F^k{}_m)\ . \tag{3}$$

The latter formulas are correct in any basis for Dirac matrices. Below we will use mainly Weyl spinor basis:

$$\gamma^a = \left| \begin{array}{cc} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{array} \right| \ , \ \ \sigma^a = (I, \sigma^j)\ , \ \ \bar{\sigma}^a = (I,\ -\sigma^j),\ \gamma^5 = \left| \begin{array}{cc} -I & 0 \\ 0 & +I \end{array} \right| \ . \tag{4}$$

With this choice, let us make 3+1 - splitting in all the formulas: in so doing we arrive at the formulas

$$G \in GL(4.C)\ , \qquad G = \left| \begin{array}{cc} k_0 + \mathbf{k}\ \vec{\sigma} & n_0 - \mathbf{n}\ \vec{\sigma} \\ -l_0 - \mathbf{l}\ \vec{\sigma} & m_0 - \mathbf{m}\ \vec{\sigma} \end{array} \right| \ , \ . \tag{5}$$

where complex 4-vector parameters (k, l, m, n) are defined by [1]:

$$k_0 = A - iB\ , \qquad k_j = a_j - ib_j\ , \qquad m_0 = A + iB\ , \qquad m_j = a_j + ib_j\ .$$
$$l_0 = B_0 - iA_0\ , \qquad l_j = B_j - iA_j\ , \qquad n_0 = B_0 + iA_0\ , \qquad n_j = B_j + iA_j\ . \tag{6}$$

In such parameters (5), the composition rule (3) will look as follows:

$$(k'', m''; n'', l'') = (k', m'; n', l')(k, m; n, l)\ ;$$
$$k''_0 = k'_0\ k_0 + \mathbf{k}'\ \mathbf{k} - n'_0\ l_0 + \mathbf{n}'\ \mathbf{l}\ ,$$
$$\mathbf{k}'' = k'_0\ \mathbf{k} + \mathbf{k}'\ k_0 + i\ \mathbf{k}' \times \mathbf{k} - n'_0\ \mathbf{l} + \mathbf{n}'\ l_0 + i\ \mathbf{n}' \times \mathbf{l}\ ,$$
$$m''_0 = m'_0\ m_0 + \mathbf{m}'\ \mathbf{m} - l'_0\ n_0 + \mathbf{l}'\ \mathbf{n}\ ,$$
$$\mathbf{m}'' = m'_0\ \mathbf{m} + \mathbf{m}'\ m_0 - i\ \mathbf{m}' \times \mathbf{m} - l'_0\ \mathbf{n} + \mathbf{l}'\ n_0 - i\ \mathbf{l}' \times \mathbf{n}\ ,$$
$$n''_0 = k'_0\ n_0 - \mathbf{k}'\ \mathbf{n} + n'_0\ m_0 + \mathbf{n}'\ \mathbf{m}\ ,$$
$$\mathbf{n}'' = k'_0\ \mathbf{n} - \mathbf{k}'\ n_0 + i\ \mathbf{k}' \times \mathbf{n} + n'_0\ \mathbf{m} + \mathbf{n}'\ m_0 - i\ \mathbf{n}' \times \mathbf{m}\ ,$$
$$l''_0 = l'_0\ k_0 + \mathbf{l}'\ \mathbf{k} + m'_0\ l_0 - \mathbf{m}'\ \mathbf{l}\ ,$$
$$\mathbf{l}'' = l'_0\ \mathbf{k} + \mathbf{l}'\ k_0 + i\ \mathbf{l}' \times \mathbf{k} + m'_0\ \mathbf{l} - \mathbf{m}'\ l_0 - i\ \mathbf{m}' \times \mathbf{l}\ . \tag{7}$$

These formulas are correct for any matrices of $GL(4.C)$. They can be looked as generalization for Fedorov's method of the vector parametrization in the theory of the Lorentz group [3] for the case of $GL(4.C)$ group.

Now let us turn to the main task: at given $G = G(k, m, n, l)$ one should find parameters of the inverse matrix:

$$G^{-1} = G(k', m', n', l') \,. \tag{8}$$

In other words, starting from

$$G(k, m, n, l) = \begin{vmatrix} +(k_0 + k_3) & +(k_1 - ik_2) & +(n_0 - n_3) & -(n_1 - in_2) \\ +(k_1 + ik_2) & +(k_0 - k_3) & -(n_1 + in_2) & +(n_0 + n_3) \\ -(l_0 + l_3) & -(l_1 - il_2) & +(m_0 - m_3) & -(m_1 - im_2) \\ -(l_1 + il_2) & -(l_0 - l_3) & -(m_1 + im_2) & +(m_0 + m_3) \end{vmatrix} , \tag{9}$$

one should calculate parameters of the inverse matrix

$$G^{-1} = D^{-1} \begin{vmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{vmatrix} , \tag{10}$$

$D = \det G$ stands for determinant of G. The problem turns to be rather complicated, the final result is (the notation is used: $(mn) = m_0 n_0 - m_1 n_1 - m_2 n_2 - m_3 n_3$, and so on.):

$$\begin{aligned}
k'_0 &= D^{-1} \, [\, k_0 \, (mm) + m_0 \, (ln) + l_0 \, (nm) - n_0 (lm) + i \, \mathbf{l} \, (\mathbf{m} \times \mathbf{n}) \,] \,, \\
\mathbf{k}' &= D^{-1} \, [\, -\mathbf{k} \, (mm) - \mathbf{m} \, (ln) - \mathbf{l} \, (nm) \; + \mathbf{n}(lm) \; + 2 \, \mathbf{l} \times (\mathbf{n} \times \mathbf{m}) \; + \\
&\qquad + i \, m_0 (\mathbf{n} \times \mathbf{l}) \; + i \, l_0 (\mathbf{n} \times \mathbf{m}) + i \, n_0 (\mathbf{l} \times \mathbf{m}) \,] \,, \\
m'_0 &= D^{-1} \, [\, k_0 \, (ln) + m_0 \, (kk) - l_0 (kn) + n_0 \, lk) \; + i \, \mathbf{n} \, (\mathbf{l} \times \mathbf{k}) \,] \,, \\
\mathbf{m}' &= D^{-1} \, [\, -\mathbf{k} \, (ln) - \mathbf{m} \, (kk) + \mathbf{l} \, (kn) - \mathbf{n} \, (kl) \; + \; 2 \, \mathbf{n} \times (\mathbf{l} \times \mathbf{k}) \; + \\
&\qquad + \, i \, n_0 \, (\mathbf{k} \times \mathbf{l}) + i \, l_0 \, (\mathbf{k} \times \mathbf{n}) + i \, k_0 \, (\mathbf{n} \times \mathbf{l}) \,] \,, \\
l'_0 &= D^{-1} \, [\, +k_0 \, (ml) - m_0 \, (kl) - l_0 \, (km) - n_0 \, (ll) \; + i \, \mathbf{m} \, (\mathbf{l} \times \mathbf{k}) \,] \,, \\
\mathbf{l}' &= D^{-1} \, [\, +\mathbf{k} \, (ml) - \mathbf{m} \, (kl) - \mathbf{l} \, (km) - \mathbf{n} \, (ll) \; + 2 \, \mathbf{m} \times (\mathbf{k} \times \mathbf{l}) \; + \\
&\qquad + \, i \, m_0 \, (\mathbf{l} \times \mathbf{k}) + i \, k_0 \, (\mathbf{l} \times \mathbf{m}) + i \, l_0 \, (\mathbf{m} \times \mathbf{k}) \,] \,, \\
n'_0 &= D^{-1} \, [\, -k_0 \, (nm) + m_0 \, (kn) - l_0 \, (nn) - n_0 \, (km) \; + \; i \, \mathbf{k} \, (\mathbf{m} \times \mathbf{n}) \,], \\
\mathbf{n}' &= D^{-1} \, [\, -\mathbf{k} \, (nm) + \mathbf{m} \, (kn) - \mathbf{l} \, (nn) - \mathbf{n} \, (km) \; + \; 2 \, \mathbf{k} \times (\mathbf{m} \times \mathbf{n}) \; + \\
&\qquad + \, ik_0 \, (\mathbf{m} \times \mathbf{n}) + im_0 \, (\mathbf{k} \times \mathbf{n}) + in_0 \, (\mathbf{m} \times \mathbf{k}) \,] \,.
\end{aligned} \tag{11}$$

Substituting eqs. (11) into equation $G^{-1} G = I$ one arrives at

$$\begin{aligned}
D = k''_0 &= k'_0 \, k_0 + \mathbf{k}' \, \mathbf{k} - n'_0 \, l_0 + \mathbf{n}' \, \mathbf{l} \,, \\
0 = \mathbf{k}'' &= k'_0 \, \mathbf{k} + \mathbf{k}' \, k_0 + i \, \mathbf{k}' \times \mathbf{k} - n'_0 \, \mathbf{l} + \mathbf{n}' \, l_0 + i \, \mathbf{n}' \times \mathbf{l} \,, \\
D = m''_0 &= m'_0 \, m_0 + \mathbf{m}' \, \mathbf{m} - l'_0 \, n_0 + \mathbf{l}' \, \mathbf{n} \,, \\
0 = \mathbf{m}'' &= m'_0 \, \mathbf{m} + \mathbf{m}' \, m_0 - i \, \mathbf{m}' \times \mathbf{m} - l'_0 \, \mathbf{n} + \mathbf{l}' \, n_0 - i \, \mathbf{l}' \times \mathbf{n} \,, \\
0 = n''_0 &= k'_0 \, n_0 - \mathbf{k}' \, \mathbf{n} + n'_0 \, m_0 + \mathbf{n}' \, \mathbf{m} \,, \\
0 = \mathbf{n}'' &= k'_0 \, \mathbf{n} - \mathbf{k}' \, n_0 + i \, \mathbf{k}' \times \mathbf{n} + n'_0 \, \mathbf{m} + \mathbf{n}' \, m_0 - i \, \mathbf{n}' \times \mathbf{m} \,, \\
0 = l''_0 &= l'_0 \, k_0 + \mathbf{l}' \, \mathbf{k} + m'_0 \, l_0 - \mathbf{m}' \, \mathbf{l} \,, \\
0 = \mathbf{l}'' &= l'_0 \, \mathbf{k} + \mathbf{l}' \, k_0 + i \, \mathbf{l}' \times \mathbf{k} + m'_0 \, \mathbf{l} - \mathbf{m}' \, l_0 - i \, \mathbf{m}' \times \mathbf{l} \,.
\end{aligned}$$

After calculation, one can prove these identities and find an explicit form for D:

$$\begin{aligned}
D &= \det G(k, m, n, l) = \\
&= (kk) \, (mm) + (ll) \, (nn) + 2 \, (mk) \, (ln) + 2 \, (lk) \, (nm) - 2 \, (nk) \, (lm) + \\
&+ 2 \, i \, [\, k_0 \, \mathbf{l}(\mathbf{m} \times \mathbf{n}) \; + \; m_0 \, \mathbf{k}(\mathbf{n} \times \mathbf{l}) \; + \; l_0 \, \mathbf{k}(\mathbf{n} \times \mathbf{m}) \; + \; n_0 \, \mathbf{l}(\mathbf{m} \times \mathbf{k}) \,] \; + \\
&\qquad + 4 (\mathbf{kn}) \, (\mathbf{ml}) - 4 (\mathbf{km}) \, (\mathbf{nl}) \,.
\end{aligned} \tag{12}$$

Let us specify several more simple sub-groups.

Variant A

Let all 0-components are real-valued, and all 3-vectors are imaginary. Performing in (7) the formal change

$$\mathbf{k} \Longrightarrow i\,\mathbf{k}\,, \qquad \mathbf{m} \Longrightarrow i\,\mathbf{m}\,, \qquad \mathbf{l} \Longrightarrow i\,\mathbf{l}\,, \qquad \mathbf{n} \Longrightarrow i\,\mathbf{n}\,,$$

$$G = \left| \begin{array}{cc} k_0 + i\,\mathbf{k}\,\vec{\sigma} & n_0 - i\,\mathbf{n}\,\vec{\sigma} \\ -l_0 - i\,\mathbf{l}\,\vec{\sigma} & m_0 - i\,\mathbf{m}\,\vec{\sigma} \end{array} \right| \,, \qquad (13)$$

the rule (7) for 16 real variables looks as follows

$$\begin{aligned}
k''_0 &= k'_0\,k_0 - \mathbf{k}'\,\mathbf{k} - n'_0\,l_0 - \mathbf{n}'\,\mathbf{l}\,,\\
\mathbf{k}'' &= k'_0\,\mathbf{k} + \mathbf{k}'\,k_0 - \ \mathbf{k}' \times \mathbf{k} - n'_0\,\mathbf{l} + \mathbf{n}'\,l_0 - \ \mathbf{n}' \times \mathbf{l}\,,\\
m''_0 &= m'_0\,m_0 - \mathbf{m}'\,\mathbf{m} - l'_0\,n_0 - \mathbf{l}'\,\mathbf{n}\,,\\
\mathbf{m}'' &= m'_0\,\mathbf{m} + \mathbf{m}'\,m_0 + \ \mathbf{m}' \times \mathbf{m} - l'_0\,\mathbf{n} + \mathbf{l}'\,n_0 + \ \mathbf{l}' \times \mathbf{n}\,,\\
n''_0 &= k'_0\,n_0 + \mathbf{k}'\,\mathbf{n} + n'_0\,m_0 - \mathbf{n}'\,\mathbf{m}\,,\\
\mathbf{n}'' &= k'_0\,\mathbf{n} - \mathbf{k}'\,n_0 - \ \mathbf{k}' \times \mathbf{n} + n'_0\,\mathbf{m} + \mathbf{n}'\,m_0 + \ \mathbf{n}' \times \mathbf{m}\,,\\
l''_0 &= l'_0\,k_0 - \mathbf{l}'\,\mathbf{k} + m'_0\,l_0 + \mathbf{m}'\,\mathbf{l}\,,\\
\mathbf{l}'' &= l'_0\,\mathbf{k} + \mathbf{l}'\,k_0 - \ \mathbf{l}' \times \mathbf{k} + m'_0\,\mathbf{l} - \mathbf{m}'\,l_0 + \mathbf{m}' \times \mathbf{l}\,.
\end{aligned} \qquad (14)$$

Correspondingly, expression for determinant (12) becomes

$$\begin{aligned}
D = [kk]\,[mm] + [ll]\,[nn] + 2\,[mk]\,[ln] + 2\,[lk]\,[nm] - 2\,[nk]\,[lm] + \\
+ 2\ [\ k_0\,\mathbf{l}(\mathbf{m} \times \mathbf{n})\ +\ m_0\,\mathbf{k}(\mathbf{n} \times \mathbf{l})\ +\ l_0\,\mathbf{k}(\mathbf{n} \times \mathbf{m})\ +\ n_0\,\mathbf{l}(\mathbf{m} \times \mathbf{k})\ ]\ + \\
+4(\mathbf{kn})\,(\mathbf{ml}) - 4(\mathbf{km})\,(\mathbf{nl})\,;
\end{aligned} \qquad (15)$$

where the notation is used: $[ab] = a_0 a_0 + a_1 a_1 + a_2 a_2 + a_3 a_3$.

Variant B

Equations (7) permit the following restrictions:

$$m_a = k^*_a\,, \qquad l_a = n^*_a\,, \qquad (16)$$

and eqs. (7)) become

$$\begin{aligned}
k''_0 &= k'_0\,k_0 + \mathbf{k}'\,\mathbf{k} - n'_0\,n^*_0 + \mathbf{n}'\,\mathbf{n}^*\,,\\
\mathbf{k}'' &= k'_0\,\mathbf{k} + \mathbf{k}'\,k_0 + i\,\mathbf{k}' \times \mathbf{k} - n'_0\,\mathbf{n}^* + \mathbf{n}'\,n^*_0 + i\,\mathbf{n}' \times \mathbf{n}^*\,,\\
n''_0 &= k'_0\,n_0 - \mathbf{k}'\,\mathbf{n} + n'_0\,k^*_0 + \mathbf{n}'\,\mathbf{k}^*\,,\\
\mathbf{n}'' &= k'_0\,\mathbf{n} - \mathbf{k}'\,n_0 + i\,\mathbf{k}' \times \mathbf{n} + n'_0\,\mathbf{k}^* + \mathbf{n}'\,k^*_0 - i\,\mathbf{n}' \times \mathbf{k}^*\,.
\end{aligned} \qquad (17)$$

Determinant D is given by

$$\begin{aligned}
D = (kk)\,(kk)^* + (nn)^*\,(nn) + 2\,(k^*k)\,(n^*n) + 2\,(n^*k)\,(nk^*) - 2\,(nk)\,(nk)^* + \\
+ 2\,i\,[\,k_0\,\mathbf{k}^*(\mathbf{n} \times \mathbf{n}^*)\ -\ k^*_0\,\mathbf{k}(\mathbf{n}^* \times \mathbf{n})\ +\ n^*_0\,\mathbf{n}(\mathbf{k} \times \mathbf{k}^*)\ -\ n_0\,\mathbf{n}^*(\mathbf{k}^* \times \mathbf{k})\ ]\ + \\
+4(\mathbf{kn})\,(\mathbf{k}^*\mathbf{n}^*) - 4(\mathbf{kk}^*)\,(\mathbf{nn}^*)\,.
\end{aligned} \qquad (18)$$

Variant C

In (13) one can impose additional restrictions:

$$m_0 = k_0 \; , \qquad l_0 = n_0 \; , \qquad \mathbf{m} = -\mathbf{k} \; , \qquad \mathbf{l} = -\mathbf{n} \; ; \tag{19}$$

at this $G(k_0, \mathbf{k}, n_0, \mathbf{n})$ looks

$$G = \left| \begin{array}{cc} (k_0 + i\, \mathbf{k}\, \vec{\sigma}) & (n_0 - i\, \mathbf{n}\, \vec{\sigma}) \\ -(n_0 - i\, \mathbf{n}\, \vec{\sigma}) & (k_0 + i\, \mathbf{k}\, \vec{\sigma}) \end{array} \right| \; ; \tag{20}$$

and composition rule is

$$\begin{aligned} k_0'' &= k_0'\, k_0 - \mathbf{k}'\, \mathbf{k} - n_0'\, n_0 + \mathbf{n}'\, \mathbf{n} \; , \\ \mathbf{k}'' &= k_0'\, \mathbf{k} + \mathbf{k}'\, k_0 - \; \mathbf{k}' \times \mathbf{k} + n_0'\, \mathbf{n} + \mathbf{n}'\, n_0 + \; \mathbf{n}' \times \mathbf{n} \; , \\ n_0'' &= k_0'\, n_0 + \mathbf{k}'\, \mathbf{n} + n_0'\, k_0 + \mathbf{n}'\, \mathbf{k} \; , \\ \mathbf{n}'' &= k_0'\, \mathbf{n} - \mathbf{k}'\, n_0 - \; \mathbf{k}' \times \mathbf{n} - n_0'\, \mathbf{k} + \mathbf{n}'\, k_0 - \; \mathbf{n}' \times \mathbf{k} \; . \end{aligned} \tag{21}$$

Determinant equals to

$$\begin{aligned} \det G = & [kk]\, [kk] + [nn]\, [nn] + \\ & + 2\, (kk)\, (nn) + 2\, (nk)\, (nk) - 2\, [nk]\, [nk]) + \\ & +4(\mathbf{kn})\, (\mathbf{kn}) - 4(\mathbf{kk})\, (\mathbf{nn}) \; . \end{aligned} \tag{22}$$

Variant D

There exist one other sub-group defined by

$$n_a = 0 \; , \qquad l_a = 0 \; , \qquad G = \left| \begin{array}{cc} (k_0 + \mathbf{k}\, \vec{\sigma}) & 0 \\ 0 & (m_0 - \mathbf{m}\, \vec{\sigma}) \end{array} \right| \; , \tag{23}$$

the composition law (7) becomes most simple:

$$\begin{aligned} k_0'' = k_0'\, k_0 + \mathbf{k}'\, \mathbf{k} \; , \qquad & \mathbf{k}'' = k_0'\, \mathbf{k} + \mathbf{k}'\, k_0 + i\, \mathbf{k}' \times \mathbf{k} \; , \\ m_0'' = m_0'\, m_0 + \mathbf{m}'\, \mathbf{m} \; , \quad & \mathbf{m}'' = m_0'\, \mathbf{m} + \mathbf{m}'\, m_0 - i\, \mathbf{m}' \times \mathbf{m} \; . \end{aligned} \tag{24}$$

as well as determinant D:

$$\det G = (kk)\, (mm) \; . \tag{25}$$

If one additionally imposes two requirements $(kk) = +1$, $(mm) = +1$, the case **D** describes spinor covering for special complex rotation group $SO(4.C)$; this case was considered in detail in [1].

It should be noted that expression (12) for determinant can be transformed to the mores short form

$$\begin{aligned} \det G = & (kk)\, (mm) + (nn)\, (ll) + 2\, [kn]\, [ml] \; - \\ & -2\, (\, k_0\, \mathbf{n} + n_0\, \mathbf{k} - i\, \mathbf{k} \times \mathbf{n}\,)\, (\, m_0\, \mathbf{l} + l_0\, \mathbf{m} + i\, \mathbf{m} \times \mathbf{l}\,) \; , \end{aligned} \tag{26}$$

which for threes cases A, B, C becomes simpler:

$$\begin{aligned} (A): \quad & \det G = [kk]\, [mm] + [nn]\, [ll] + 2\, (kn)\, (ml) \; + \\ & +2\, (\, k_0\, \mathbf{n} + n_0\, \mathbf{k} + \mathbf{k} \times \mathbf{n}\,)\, (\, m_0\, \mathbf{l} + l_0\, \mathbf{m} - \mathbf{m} \times \mathbf{l}\,) \; , \\ (B): \quad & \det G = (kk)\, (k^* k^*) + (nn)\, (n^* n^*) + 2\, [kn]\, [k^* n^*] \; - \\ & -2\, (\, k_0\, \mathbf{n} + n_0\, \mathbf{k} - i\, \mathbf{k} \times \mathbf{n}\,)\, (\, k_0^*\, \mathbf{n}^* + n_0^*\, \mathbf{k}^* + i\, \mathbf{k}^* \times \mathbf{n}^*\,) \; , \\ (C): \quad & \det G = [kk]^2 + [nn]^2 + 2\, (kn)^2 \; - 2\, (\, k_0\, \mathbf{n} + n_0\, \mathbf{k} + \mathbf{k} \times \mathbf{n}\,)^2 \; . \end{aligned} \tag{27}$$

2. Unitarity condition

Now let us turn to considering the unitary group $SU(4)$. One should specify the requirement of unitarity $G^{+} = G^{-1}$ in vector parametrization. Taking into account the formulas

$$G = \left| \begin{array}{cc} k_0 + \mathbf{k}\vec{\sigma} & n_0 - \mathbf{n}\,\vec{\sigma} \\ -l_0 - \mathbf{l}\vec{\sigma} & m_0 - \mathbf{m}\vec{\sigma} \end{array} \right| , G^{+} = \left| \begin{array}{cc} k_0^* + \mathbf{k}^*\vec{\sigma} & -l_0^* - \mathbf{l}^*\vec{\sigma} \\ n_0^* - \mathbf{n}^*\vec{\sigma} & m_0^* - \mathbf{m}^*\vec{\sigma} \end{array} \right| , G^{-1} = \left| \begin{array}{cc} k_0' + \mathbf{k}'\vec{\sigma} & n_0' - \mathbf{n}'\vec{\sigma} \\ -l_0' - \mathbf{l}'\vec{\sigma} & m_0' - \mathbf{m}'\vec{\sigma} \end{array} \right| ,$$

which can be represented differently:

$$\begin{array}{rl} G = & G(k_0,\ \mathbf{k};\ \ m_0,\ \mathbf{m};\ \ n_0,\ \mathbf{n},\ \ l_0,\ \mathbf{l}\ ) , \\ G^{+} = & G(k_0^*,\ \mathbf{k}^*;\ m_0^*,\ \mathbf{m}^*;\ -l_0^*,\ \mathbf{l}^*,\ -n_0^*,\ \mathbf{n}^*) , \\ G^{-1} = & G(k_0',\ \mathbf{k}';\ m_0',\ \mathbf{m}';\ \ n_0',\ \mathbf{n}',\ \ l_0',\ \mathbf{l}'\ ) . \end{array} \tag{28}$$

Therefore, relation $G^{+} = G^{-1}$ is equivalent to the system of equations:

$$\begin{array}{llll} k_0^* = k_0' , & \mathbf{k}^* = \mathbf{k}' , & m_0^* = m_0' , & \mathbf{m}^* = \mathbf{m}' , \\ -l_0^* = n_0' , & \mathbf{l}^* = \mathbf{n}' , & -n_0^* = l_0' , & \mathbf{n}^* = \mathbf{l}' . \end{array} \tag{29}$$

With the use of expression for parameters of the inverse matrix with additional restriction $\det G = +1$; from (29) it follows

$$\begin{array}{r} k_0^* = \ +k_0(mm)\ +m_0(ln)\ +l_0(nm)\ -n_0(lm)\ +i\mathbf{l}(\mathbf{m}\times\mathbf{n}), \\ m_0^* = \ +m_0(kk)\ +k_0(nl)\ +n_0(lk)\ -l_0(nk)\ -i\mathbf{n}(\mathbf{k}\times\mathbf{l}), \\ \mathbf{k}^* = -\mathbf{k}(mm) - \mathbf{m}(ln) - \mathbf{l}(nm) + \mathbf{n}(lm) + 2\mathbf{l}\times(\mathbf{n}\times\mathbf{m}) + \\ +im_0(\mathbf{n}\times\mathbf{l}) + il_0(\mathbf{n}\times\mathbf{m}) + in_0(\mathbf{l}\times\mathbf{m}), \\ \mathbf{m}^* = -\mathbf{m}(kk) - \mathbf{k}(nl) - \mathbf{n}(lk) + \mathbf{l}(nk) + 2\mathbf{n}\times(\mathbf{l}\times\mathbf{k}) - \\ -ik_0(\mathbf{l}\times\mathbf{n}) - in_0(\mathbf{l}\times\mathbf{k}) - il_0(\mathbf{n}\times\mathbf{k}) , \\ l_0^* = \ +k_0(nm)\ -m_0(kn)\ +l_0(nn)\ +n_0(km)\ +i\mathbf{k}(\mathbf{n}\times\mathbf{m}) , \\ n_0^* = \ +m_0(lk)\ -k_0(ml)\ +n_0(ll)\ +l_0(mk)\ -i\mathbf{m}(\mathbf{l}\times\mathbf{k}) , \\ \mathbf{l}^* = -\mathbf{k}(nm) + \mathbf{m}(kn) - \mathbf{l}(nn) - \mathbf{n}(km) + 2\mathbf{k}\times(\mathbf{m}\times\mathbf{n}) + \\ +ik_0(\mathbf{m}\times\mathbf{n}) + im_0(\mathbf{k}\times\mathbf{n}) + in_0(\mathbf{m}\times\mathbf{k}) , \\ \mathbf{n}^* = -\mathbf{m}(kl) + \mathbf{k}(ml) - \mathbf{n}(ll) - \mathbf{l}(mk) + 2\mathbf{m}\times(\mathbf{k}\times\mathbf{l}) - im_0(\mathbf{k}\times\mathbf{l}) - \\ -ik_0(\mathbf{m}\times\mathbf{l}) - il_0(\mathbf{k}\times\mathbf{m}). \end{array} \tag{30}$$

Thus, the known form for parameters of the inverse matrix G^{-1} have made possible to write easily relations (30) representing the unitarity condition. Here there are 16 equations for 16 variables; evidently, they all are not independent. Let us write down several more simple variants.

Variant A

With formal change

$$\mathbf{k} \Longrightarrow i\,\mathbf{k} , \qquad \mathbf{m} \Longrightarrow i\,\mathbf{m} , \qquad \mathbf{l} \Longrightarrow i\,\mathbf{l} , \qquad \mathbf{n} \Longrightarrow i\,\mathbf{n} , \tag{31}$$

from (30) it follows

$$
\begin{aligned}
k_0 &= +k_0[mm] \ +m_0[ln] \ +l_0[nm] \ -n_0[lm] \ +\mathbf{l}(\mathbf{m}\times\mathbf{n}) \, , \\
m_0 &= +m_0[kk] \ +k_0[nl] \ +n_0[lk] \ -l_0[nk] \ -\mathbf{n}(\mathbf{k}\times\mathbf{l}) \, , \\
\mathbf{k} &= \mathbf{k}[mm] + \mathbf{m}[ln] + \mathbf{l}[nm] - \mathbf{n}[lm] + 2\mathbf{l}\times(\mathbf{n}\times\mathbf{m}) + \\
&\quad + m_0(\mathbf{n}\times\mathbf{l}) + l_0(\mathbf{n}\times\mathbf{m}) + n_0(\mathbf{l}\times\mathbf{m}) \, , \\
\mathbf{m} &= +\mathbf{m}[kk] + \mathbf{k}[nl] + \mathbf{n}[lk] - \mathbf{l}[nk] + 2\mathbf{n}\times(\mathbf{l}\times\mathbf{k}) - \\
&\quad - k_0(\mathbf{l}\times\mathbf{n}) - n_0(\mathbf{l}\times\mathbf{k}) - l_0(\mathbf{n}\times\mathbf{k}) \, , \\
l_0 &= +k_0[nm] \ -m_0[kn] \ +l_0[nn] \ +n_0[km] \ +\mathbf{k}(\mathbf{n}\times\mathbf{m}) \, , \\
n_0 &= +m_0[lk] \ -k_0[ml] \ +n_0[ll] \ +l_0[mk] \ -\mathbf{m}(\mathbf{l}\times\mathbf{k}) \, , \\
\mathbf{l} &= +\mathbf{k}[nm] - \mathbf{m}[kn] + \mathbf{l}[nn] + \mathbf{n}[km] + 2\mathbf{k}\times(\mathbf{m}\times\mathbf{n}) + \\
&\quad + k_0(\mathbf{m}\times\mathbf{n}) + m_0(\mathbf{k}\times\mathbf{n}) + n_0(\mathbf{m}\times\mathbf{k}) \, , \\
\mathbf{n} &= +\mathbf{m}[kl] - \mathbf{k}[ml] + \mathbf{n}[ll] + \mathbf{l}[mk] + 2\mathbf{m}\times(\mathbf{k}\times\mathbf{l}) - \\
&\quad - m_0(\mathbf{k}\times\mathbf{l}) - k_0(\mathbf{m}\times\mathbf{l}) - l_0(\mathbf{k}\times\mathbf{m}) .
\end{aligned} \tag{32}
$$

Here there are 16 equations for 16 real-valued variables.

Variant B

Let

$$
\begin{aligned}
m_0 &= k_0^* \, , \qquad \mathbf{m} = \mathbf{k}^* \, , \qquad l_0 = n_0^* \, , \qquad \mathbf{l} = \mathbf{n}^* \, , \\
k_0 &= m_0^* \, , \qquad \mathbf{k} = \mathbf{m}^* \, , \qquad n_0 = l_0^* \, , \qquad \mathbf{n} = \mathbf{l}^* \, ,
\end{aligned} \tag{33}
$$

or symbolically $m = k^*$, $l = n^*$. Unitarity relations become

$$
\begin{aligned}
k_0^* &= +k_0(k^*k^*) + k_0^*(n^*n) + n_0^*(nk^*) - n_0(n^*k^*) + i\mathbf{n}^*(\mathbf{k}^*\times\mathbf{n}) \, , \\
\mathbf{k}^* &= -\mathbf{k}(k^*k^*) - \mathbf{k}^*(n^*n) - \mathbf{n}^*(nk^*) + \mathbf{n}(n^*k^*) + \\
&\quad +2\mathbf{n}^*\times(\mathbf{n}\times\mathbf{k}^*) + ik_0^*(\mathbf{n}\times\mathbf{n}^*) + in_0^*(\mathbf{n}\times\mathbf{k}^*) + in_0(\mathbf{l}\times\mathbf{m}) \, , \\
n_0^* &= +k_0^*(n^*k) - k_0(k^*n^*) + n_0(n^*n^*) + n_0^*(k^*k) - i\mathbf{k}^*(\mathbf{n}^*\times\mathbf{k}) \, , \\
\mathbf{n}^* &= -\mathbf{k}^*(kn^*) + \mathbf{k}(k^*n^*) - \mathbf{n}(n^*n^*) - \mathbf{n}^*(k^*k) \ + \\
&\quad +2\mathbf{k}^*\times(\mathbf{k}\times\mathbf{n}^*) - ik_0^*(\mathbf{k}\times\mathbf{n}^*) - ik_0(\mathbf{k}^*\times\mathbf{n}^*) - in_0^*(\mathbf{k}\times\mathbf{k}^*) \, ,
\end{aligned} \tag{34}
$$

and 8 conjugated ones

$$
\begin{aligned}
k_0 &= +k_0^*(kk) \ +k_0(nn^*) \ +n_0(n^*k) \ -n_0^*(nk) \ -i\mathbf{n}(\mathbf{k}\times\mathbf{n}^*) \, , \\
\mathbf{k} &= -\mathbf{k}^*(kk) - \mathbf{k}(nn^*) - \mathbf{n}(n^*k) + \mathbf{n}^*(nk) + \\
&\quad +2\mathbf{n}\times(\mathbf{n}^*\times\mathbf{k}) - ik_0(\mathbf{n}^*\times\mathbf{n}) - in_0(\mathbf{n}^*\times\mathbf{k}) - in_0^*(\mathbf{n}\times\mathbf{k}) \, , \\
n_0 &= +k_0(nk^*) - k_0^*(kn) + n_0^*(nn) + n_0(kk^*) + i\mathbf{k}(\mathbf{n}\times\mathbf{k}^*) \, , \\
\mathbf{n} &= -\mathbf{k}(nk^*) + \mathbf{k}^*(kn) - \mathbf{n}^*(nn) - \mathbf{n}(kk^*) \ + \\
&\quad +2\mathbf{k}\times(\mathbf{k}^*\times\mathbf{n}) + ik_0(\mathbf{k}^*\times\mathbf{n}) + ik_0^*(\mathbf{k}\times\mathbf{n}) + in_0(\mathbf{k}^*\times\mathbf{k}) .
\end{aligned} \tag{35}
$$

It may be noted that latter relations are greatly simplified when $n = 0$, or when $k = 0$. In the firs place, let us consider the case $n = 0$:

$$
k_0^* = +k_0(k^*k^*) \, , \qquad \mathbf{k}^* = -\mathbf{k}(k^*k^*) \, . \tag{36}
$$

Taking in mind the identity

$$
\det G = (kk)\,(kk)^* = +1 \, , \implies \qquad (kk) = +1 \, , \qquad (kk)^* = +1 \, .
$$

we arrive at $k_0^* = +k_0$, $\mathbf{k}^* = -\mathbf{k}$. In has sense to introduce the real-valued vector c_a:

$$k_0^* = +k_0 = c_0 \; , \qquad \mathbf{k}^* = -\mathbf{k} : \qquad \mathbf{k} = i\mathbf{c} \; , \tag{37}$$

then matrix G is

$$G(k, m = k^*, 0, 0) = \left| \begin{array}{cc} c_0 + i\mathbf{c}\,\vec{\sigma} & 0 \\ 0 & c_0 - i\mathbf{c}\,\vec{\sigma} \end{array} \right| \sim SU(2) \; . \tag{38}$$

Another possibility is realized when $k = 0$, then

$$n_0^* = +n_0 (nn)^* \; , \qquad \mathbf{n}^* = -\mathbf{n}(nn)^* \; . \tag{39}$$

With the use of identity

$$\det G = (nn)\;(nn)^* = +1 \; , \qquad \Longrightarrow \qquad (nn) = +1 \; , \qquad (nn)^* = +1 \; ,$$

we get

$$n_0^* = +n_0 = c_0 \; , \qquad \mathbf{n}^* = -\mathbf{n} \; , \qquad \mathbf{n} \equiv i\;\mathbf{c} \tag{40}$$

Corresponding matrices $G(0, 0, n, l = n^*)$ make up special set of unitary matrices

$$G = \left| \begin{array}{cc} 0 & c_0 - i\mathbf{c}\,\vec{\sigma} \\ -(c_0 + i\mathbf{c}\,\vec{\sigma}) & 0 \end{array} \right| \; , \qquad G^+ = \left| \begin{array}{cc} 0 & -(c_0 - i\mathbf{c}\,\vec{\sigma}) \\ (c_0 + i\mathbf{c}\,\vec{\sigma}) & 0 \end{array} \right| \; . \tag{41}$$

However, it must be noted that these matrices (41) do not privide us with any sub-group because we have only

$$G^+ G = 1 \; , \qquad G^2 = I \;\; : \qquad \Longrightarrow \qquad G^+ = G^{-1} = -G. \tag{42}$$

Variant C

Now in eqs. (31) one should take

$$m_0 = k_0 \; , \qquad l_0 = n_0 \; , \qquad \mathbf{m} = -\mathbf{k} \; , \qquad \mathbf{l} = -\mathbf{n} \; , \tag{43}$$

then

$$\begin{aligned} k_0 &= +k_0[kk] + k_0(nn) + n_0(nk) - n_0[nk] \; , \\ \mathbf{k} &= \mathbf{k}[kk] - \mathbf{k}(nn) - \mathbf{n}(nk) - \mathbf{n}[nk] + 2\mathbf{n} \times (\mathbf{n} \times \mathbf{k}) \; , \\ n_0 &= +k_0(nk) - k_0[kn] + n_0[nn] + n_0(kk) \; , \\ \mathbf{n} &= -\mathbf{k}(kn) - \mathbf{k}[kn] + \mathbf{n}[nn] - \mathbf{n}(kk) + 2\mathbf{k} \times (\mathbf{k} \times \mathbf{n}) \; . \end{aligned} \tag{44}$$

3. 2-parametric sub-groups in $SU(4)$

To be certain in correctness of the above produced equations of unitarity, one should try to solve them at least in several most simple particular cases. For instance, let us specify equations (44) for transformations from sub-group G_1, that is when $k = (k_0, k_1, 0, 0)$, $n = (n_0, n_1, 0, 0)$:

$$\begin{aligned} k_0 &= +k_0[kk] + k_0(nn) + n_0(nk) - n_0[nk] \; , \\ k_1 &= +k_1[kk] - k_1(nn) - n_1(nk) - n_1[nk] \; , \\ n_0 &= +k_0(nk) - k_0[kn] + n_0[nn] + n_0(kk) \; , \\ n_1 &= -k_1(kn) - k_1[kn] + n_1[nn] - n_1(kk) \; . \end{aligned} \tag{45}$$

Here there are four non-linear equations for four real variables. It may be noted that eqs. (2.1) can be looked as two eigen-value problems in two dimensional space (with eigen-value +1):

$$\left| \begin{array}{cc} (k_0^2+n_0^2)-1+(k_1^2-n_1^2) & -2n_1k_1 \\ -2n_1k_1 & (k_0^2+n_0^2)-1-(k_1^2-n_1^2) \end{array} \right| \left| \begin{array}{c} k_0 \\ n_0 \end{array} \right| = \left| \begin{array}{c} 0 \\ 0 \end{array} \right| , \qquad (46)$$

$$\left| \begin{array}{cc} (k_1^2+n_1^2)-1+(k_0^2-n_0^2) & -2n_0k_0 \\ -2n_0k_0 & (k_1^2+n_1^2)-1-(k_0^2-n_0^2) \end{array} \right| \left| \begin{array}{c} k_1 \\ n_1 \end{array} \right| = \left| \begin{array}{c} 0 \\ 0 \end{array} \right| . \qquad (47)$$

The determinants in both problems must be equated to zero:

$$[(k_0^2+n_0^2)-1]^2-(k_1^2-n_1^2)^2-4n_1^2k_1^2=0 , \qquad [(k_1^2+n_1^2)-1]^2-(k_0^2-n_0^2)^2-4n_0^2k_0^2=0 ,$$

or

$$[(k_0^2+n_0^2)-1]^2-(k_1^2+n_1^2)^2=0 , \qquad [(k_1^2+n_1^2)-1]^2-(k_0^2+n_0^2)^2=0 . \qquad (48)$$

The latter equations may be rewritten in factorized form:

$$[(k_0^2+n_0^2)-1-(k_1^2+n_1^2)]\,[(k_0^2+n_0^2)-1+(k_1^2+n_1^2)]=0 ,$$
$$[(k_1^2+n_1^2)-1-(k_0^2+n_0^2)]\,[(k_1^2+n_1^2)-1+(k_0^2+n_0^2)]=0 . \qquad (49)$$

They have the structure: $AC=0, \;\; BC=0$. There arise four different cases.

(1) Let $C=0$, then

$$k_0^2+n_0^2+k_1^2+n_1^2=+1 . \qquad (50)$$

(2) Now, let $A=0, \; B=0$, but a contradiction arises: $A+B=0 , \qquad A+B=-2$.

(3)-(4) There are two simples cases:

$$A=0, \; C=0 : \qquad k_0^2+n_0^2=1, k_1=0, n_1=0 ; \qquad (51)$$
$$B=0, \; C=0 : \qquad k_1^2+n_1^2=1, k_0=0, n_0=0 ; \qquad (52)$$

Evidently, (51) and (52) can be looked as particular cases of the above variant (50). Now, one should take into account additional relation $\det G=+1$

$$\det G=[kk]\,[kk]+[nn]\,[nn]+2\,(kk)\,(nn)\; +$$
$$+\,2\,(nk)\,(nk)-2\,[nk]\,[nk])+4(\mathbf{kn})\,(\mathbf{kn})-4(\mathbf{kk})\,(\mathbf{nn}) ,$$

which can be transformed to

$$\det G=(k_0^2+k_1^2+n_0^2+n_1^2)^2-4(k_1n_0+k_0n_1)^2 \;=+1 . \qquad (53)$$

Both eqs. (50) and (53) are to be satisfied:

$$\left\{ \begin{array}{l} (k_0^2+n_0^2+k_1^2+n_1^2)=1 , \\ (k_0^2+k_1^2+n_0^2+n_1^2)^2-1-4(k_1n_0+k_0n_1)^2 \;=0 ; \end{array} \right. \qquad (54)$$

from where it follows

$$k_1n_0+k_0n_1=0 , \qquad k_0^2+n_0^2+k_1^2+n_1^2=+1 , \qquad (55)$$

they specify a 2-parametric unitary sub-group in $SU(4)$:

Variant C_1 :

$$G_1^+ = G_1^{-1} \; , \qquad \det G_1 = +1 \; ,$$

$$G_1 = \left| \begin{array}{cc} k_0 + i\,k_1\,\sigma^1 & n_0 - i\,n_1\,\sigma^1 \\ -n_0 + i\,n_1\,\sigma^1 & k_0 + i\,k_1\,\sigma^1 \end{array} \right| = \left| \begin{array}{cccc} k_0 & ik_1 & n_0 & -in_1 \\ ik_1 & k_0 & -in_1 & n_0 \\ -n_0 & in_1 & k_0 & ik_1 \\ in_1 & -n_0 & ik_1 & k_0 \end{array} \right| . \qquad (56)$$

Two analogous examples are possible:

Variant C_2 :

$$G_2^+ = G_2^{-1} \; , \qquad \det G_2 = +1 \; ,$$
$$k_2 n_0 + k_0 n_2 = 0 \; , k_0^2 + n_0^2 + k_2^2 + n_2^2 = +1 \; ,$$

$$G_2 = \left| \begin{array}{cc} k_0 + i\,k_2\,\sigma^2 & n_0 - i\,n_2\,\sigma^2 \\ -n_0 + i\,n_2\,\sigma^2 & k_0 + i\,k_2\,\sigma^2 \end{array} \right| = \left| \begin{array}{cccc} k_0 & k_2 & n_0 & -n_2 \\ -k_2 & k_0 & n_2 & n_0 \\ -n_0 & n_2 & k_0 & k_2 \\ -n_2 & -n_0 & -k_2 & k_0 \end{array} \right| . \qquad (57)$$

Variant C_3

$$G_3^+ = G_3^{-1} \; , \qquad \det G_3 = +1 \; ,$$
$$k_3 n_0 + k_0 n_3 = 0 \; , k_0^2 + n_0^2 + k_3^2 + n_3^2 = +1 \; ,$$

$$G_3 = \left| \begin{array}{cccc} (k_0 + ik_3) & 0 & (n_0 - in_3) & 0 \\ 0 & (k_0 - ik_3) & 0 & (n_0 + in_3) \\ -(n_0 - in_3) & 0 & (k_0 + ik_3) & 0 \\ 0 & -(n_0 + in_3) & 0 & (k_0 - ik_3) \end{array} \right| . \qquad (58)$$

Let us consider the latter subgroup in some details. The multiplication law for parameters is

$$k''_0 = k'_0\,k_0 - k'_3\,k_3 - n'_0\,n_0 + n'_3\,n_3 \; , \qquad k''_3 = k'_0\,k_3 + k'_3\,k_0 + n'_0\,n_3 + n'_3\,n_0 \; ,$$
$$n''_0 = k'_0\,n_0 + k'_3\,n_3 + n'_0\,k_0 + n'_3\,k_3 \; , \qquad n''_3 = k'_0\,n_3 - k'_3\,n_0 - n'_0\,k_3 + n'_3\,k_0 \; . \qquad (59)$$

These formulas for two particular cases — see (51) and (52)) give

$$\{G_3'^0\} : \qquad k_3^2 + n_3^2 = 1 \; , \qquad k_0 = 0, \; n_0 = 0 \; ,$$
$$k''_0 = -k'_3\,k_3 + n'_3\,n_3 \; , \qquad k''_3 = 0 \; , \qquad (60)$$
$$n''_0 = +k'_3\,n_3 + n'_3\,k_3 \; , \qquad n''_3 = 0 \; ,$$

$$\{G^0\} : \qquad k_0^2 + n_0^2 = 1 \; , \qquad k_3 = 0, \; n_3 = 0 \; ,$$
$$k''_0 = k'_0\,k_0 - n'_0\,n_0 \; , \qquad (61)$$
$$n''_0 = k'_0\,n_0 + n'_0\,k_0 \; .$$

Therefore, multiplying any two elements from $G_3'^0$ does not lead us to any element from $G_3'^0$, instead belonging to G^0: $G_3^{0'}\; G_3^0 \;\; \in \;\; G^0$. Similar result would be achieved for G_1 and G_2: $G_1^{0'}\; G_1^0 \;\; \in \;\; G^0$, $G_2^{0'}\; G_2^0 \;\; \in \;\; G^0$. In the sub-group given by (61) one can easily see the structure of the 1-parametric Abelian group:

$$k_0 = \cos\alpha \; , \qquad n_0 = \sin\alpha \; ,$$

$$G^0(\alpha) = \left| \begin{array}{cccc} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha \end{array} \right| , \qquad \alpha \in [0, \; 2\pi] \; . \qquad (62)$$

In the same manner, similar curvilinear parametrization can be readily produced for 2-parametric groups (56)-(58). For definiteness, for sub-group G_3 such coordinates are given by

$$
\begin{aligned}
k_0 = \cos\alpha\cos\rho \; , \qquad k_3 = \cos\alpha\sin\rho \; , \\
n_0 = \sin\alpha\cos\rho \; , \qquad -n_3 = \sin\alpha\sin\rho \; , \quad \alpha \in [0,\; 2\pi] \; ,
\end{aligned}
\tag{63}
$$

and matrix G_3 is

$$
G_3(\rho,\alpha) = \left| \begin{array}{cccc} \cos\alpha\; e^{i\rho} & 0 & \sin\alpha\; e^{i\rho} & 0 \\ 0 & \cos\alpha\; e^{-i\rho} & 0 & \sin\alpha\; e^{-i\rho} \\ -\sin\alpha\; e^{i\rho} & 0 & \cos\alpha\; e^{i\rho} & 0 \\ 0 & -\sin\alpha\; e^{-i\rho} & 0 & \cos\alpha\; e^{-i\rho} \end{array} \right| . \tag{64}
$$

One may note that eq. (64) at $\rho = 0$ will coincide with $G^0(\alpha)$ in (62): $G_3(\rho = 0, \alpha) = G^0(\alpha)$. Similar curvilinear parametrization may be introduced for two other sub-groups, G_1 and G_2.

One could try to obtain more general result just changing real valued curvilinear coordinates on complex. However it is easily verified that it is not the case: through that change though there arise sub-groups but they are not unitary. Indeed, let the matrix (2.16) be complex: then unitarity condition gives

$$
\cos\alpha\cos\alpha^* + \sin\alpha\sin\alpha^* = 1 \; , \qquad -\cos\alpha\sin\alpha^* + \sin\alpha\cos\alpha^* = 0 \; . \tag{65}
$$

These two equations can be satisfied only by a real valued α. In the same manner, the the formal change $G_1, G_2, G_3 \implies G_1^C, G_2^C, G_3^C$ again provides us with non-unitary sub-groups.

4. 4-parametric solution of the unitarity condition

Let us turn again to the sub-group in $GL(4.C)$ given by

$$
G = \left| \begin{array}{cc} k_0 + \mathbf{k}\vec{\sigma} & n_0 - \mathbf{n}\,\vec{\sigma} \\ -l_0 - \mathbf{l}\vec{\sigma} & m_0 - \mathbf{m}\vec{\sigma} \end{array} \right| , \tag{66}
$$

when the unitarity equation look as follows:

$$
\begin{aligned}
k_0 &= +k_0[kk] + k_0(nn) + n_0(nk) - n_0[nk] \; , \\
n_0 &= +k_0(nk) - k_0[kn] + n_0[nn] + n_0(kk) \; , \\
\mathbf{k} &= \mathbf{k}[kk] - \mathbf{k}(nn) - \mathbf{n}(nk) - \mathbf{n}[nk] + 2\mathbf{n} \times (\mathbf{n} \times \mathbf{k}) \; , \\
\mathbf{n} &= -\mathbf{k}(kn) - \mathbf{k}[kn] + \mathbf{n}[nn] - \mathbf{n}(kk) + 2\mathbf{k} \times (\mathbf{k} \times \mathbf{n}) \; .
\end{aligned}
\tag{67}
$$

They can be rewritten as for eigen-value problems:

$$
\left| \begin{array}{cc} [kk] + (nn) & (nk) - [nk] \\ (nk) - [nk] & (kk) + [nn] \end{array} \right| \left| \begin{array}{c} k_0 \\ n_0 \end{array} \right| = (+1) \left| \begin{array}{c} k_0 \\ n_0 \end{array} \right| , \tag{68}
$$

$$
\begin{aligned}
&\left| \begin{array}{cc} +([kk] - [nn]) & -2(nk) \\ -2(nk) & -([kk] - [nn]) \end{array} \right| \left| \begin{array}{c} k_1 \\ n_1 \end{array} \right| = (+1) \left| \begin{array}{c} k_1 \\ n_1 \end{array} \right| , \\
&\left| \begin{array}{cc} +([kk] - [nn]) & -2(nk) \\ -2(nk) & -([kk] - [nn]) \end{array} \right| \left| \begin{array}{c} k_2 \\ n_2 \end{array} \right| = (+1) \left| \begin{array}{c} k_2 \\ n_2 \end{array} \right| , \\
&\left| \begin{array}{cc} +([kk] - [nn]) & -2(nk) \\ -2(nk) & -([kk] - [nn]) \end{array} \right| \left| \begin{array}{c} k_3 \\ n_3 \end{array} \right| = (+1) \left| \begin{array}{c} k_3 \\ n_3 \end{array} \right| .
\end{aligned}
\tag{69}
$$

These equations have the same structure

$$\left| \begin{array}{cc} A & C \\ C & B \end{array} \right| \left| \begin{array}{c} Z_1 \\ Z_2 \end{array} \right| = \lambda \left| \begin{array}{c} Z_1 \\ Z_2 \end{array} \right| , \tag{70}$$

where $\lambda = +1$. Non-trivial solutions may exist only if

$$\det \left| \begin{array}{cc} A-\lambda & C \\ C & B-\lambda \end{array} \right| = 0 , \tag{71}$$

which gives two different eigen values

$$\lambda_1 = \frac{A + B + \sqrt{(A-B)^2 + 4C^2}}{2} , \qquad \lambda_2 = \frac{A + B + \sqrt{(A-B)^2 + 4C^2}}{2} . \tag{72}$$

In explicit form, eqs. the problem (68) looks as follows:

$$\begin{array}{c} \left| \begin{array}{cc} A & C \\ C & B \end{array} \right| \left| \begin{array}{c} k_0 \\ n_0 \end{array} \right| = \lambda \left| \begin{array}{c} k_0 \\ n_0 \end{array} \right| , \\ A = (k_0^2 + n_0^2) + (\mathbf{k}^2 - \mathbf{n}^2) , \\ B = (k_0^2 + n_0^2) - (\mathbf{k}^2 - \mathbf{n}^2) , \qquad C = -2\, \mathbf{kn} , \\ \lambda_1 = (k_0^2 + n_0^2) + \sqrt{(\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{kn})^2} , \\ \lambda_2 = (k_0^2 + n_0^2) - \sqrt{(\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{kn})^2} . \end{array} \tag{73}$$

The eigen-value $\lambda = +1$ might be constructed in two ways:

$$\begin{array}{ll} \lambda_1 = +1 , & k_0^2 + n_0^2 = 1 - \sqrt{(\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{kn})^2} , \\ \lambda_2 = +1 , & k_0^2 + n_0^2 = 1 + \sqrt{(\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{kn})^2} . \end{array} \tag{74}$$

These two relations (74) are equivalent to

$$(1 - k_0^2 - n_0^2)^2 = (\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{kn})^2 .$$

Thus, the problem (73) has two different types (see (73)):

$$\begin{array}{l} \underline{\text{Type } I :} \qquad (A-1)\, k_0 + C\ \ n_0 = 0 , \qquad C\ k_0 + (B-1)\, n_0 = 0 , \\ k_0^2 + n_0^2 = 1 - \sqrt{(\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{kn})^2} , \\ k_0^2 + n_0^2 < +1 , \qquad (\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{kn})^2 < +1 ; \end{array} \tag{75}$$

$$\begin{array}{l} \underline{\text{Type } II :} \qquad (A-1)\, k_0 + C\ \ n_0 = 0 , \qquad C\ k_0 + (B-1)\, n_0 = 0 , \\ k_0^2 + n_0^2 = 1 + \sqrt{(\mathbf{k}^2 - \mathbf{n}^2)^2 + 4(\mathbf{kn})^2} , \\ k_0^2 + n_0^2 > +1 . \end{array} \tag{76}$$

Now let us turn to eqs. (69). They have the form

$$\begin{array}{c} \left| \begin{array}{cc} A & C \\ C & -A \end{array} \right| \left| \begin{array}{c} k_i \\ n_i \end{array} \right| = \lambda \left| \begin{array}{c} k_i \\ n_i \end{array} \right| , \qquad i = 1, 2, 3 , \\ A = k_0^2 + \mathbf{k}^2 - n_0^2 - \mathbf{n}^2 , \qquad B = -A , \qquad C = -2(k_0 n_0 - \mathbf{kn}) , \\ \lambda_1 = +\sqrt{(k_0^2 + \mathbf{k}^2 - n_0^2 - \mathbf{n}^2)^2 + 4(k_0 n_0 - \mathbf{kn})^2} , \\ \lambda_2 = -\sqrt{(k_0^2 + \mathbf{k}^2 - n_0^2 - \mathbf{n}^2)^2 + 4(k_0 n_0 - \mathbf{kn})^2} . \end{array} \tag{77}$$

Because we are interested only in positive eigen-value]$lambda = +1$, we must use only one possibility $\lambda = +1 = \lambda_1$, so that

$$(A-1)\,\mathbf{k} + C\;\;\mathbf{n} = 0\,, \qquad C\,\mathbf{k} - (A+1)\,\mathbf{n} = 0\,, \qquad (78)$$
$$1 = (k_0^2 + \mathbf{k}^2 - n_0^2 - \mathbf{n}^2)^2 + 4(k_0 n_0 - \mathbf{kn})^2\,.$$

Vector condition in (78) say that $\mathbf{k}$ and $\mathbf{n}$ are (anti) collinear:

$$\mathbf{k} = K\,\mathbf{e}\,, \qquad \mathbf{n} = N\,\mathbf{e}\,, \qquad \mathbf{e}^2 = 1\,,\ \mathbf{e} \in S_2\,; \qquad (79)$$

so that (78) give

$$\begin{gathered} (A-1)\,K + C\;\;N = 0\,, \qquad C\,K - (A+1)\,N = 0\,, \\ 1 = (k_0^2 + K^2 - n_0^2 - N^2)^2 + 4(k_0 n_0 - KN)^2\,, \\ A = k_0^2 + K^2 - n_0^2 - N^2\,,\ \ C = -2(k_0 n_0 - KN)\,. \end{gathered} \qquad (80)$$

With notation (79), eqs. (75) – (76) take the form

$$\begin{aligned} \underline{\text{Type } I:} \qquad & (A-1)\,k_0 + C\;\;n_0 = 0\,, \\ & C\,k_0 + (B-1)\,n_0 = 0\,, \\ & k_0^2 + n_0^2 = 1 - (K^2 + N^2)\,, \qquad (81) \\ \underline{\text{Type } II:} \qquad & (A-1)\,k_0 + C\;\;n_0 = 0\,, \\ & C\,k_0 + (B-1)\,n_0 = 0\,, \\ & k_0^2 + n_0^2 = 1 + (K^2 + N^2)\,, \qquad (82) \end{aligned}$$

where

$$A = (k_0^2 + n_0^2) + (K^2 - N^2)\,, \qquad B = (k_0^2 + n_0^2) - (K^2 - N^2)\,, \qquad C = -2\,KN\,. \qquad (83)$$

Therefore, we have 8 variables $\mathbf{e}$, k_0, n_0, K, N and the set of equations, (80 – (83) for them. Its solving turns to be rather involving, so let us formulate only the final result:

$$\begin{gathered} k_0, \qquad \mathbf{k} = K\mathbf{e}, \qquad n_0, \qquad \mathbf{n} = N\mathbf{e}\,, \\ k_0^2 + K^2 + n_0^2 + N^2 = +1\,, \\ k_0 N + n_0 K = 0\,, \end{gathered} \qquad (84)$$

$$G = \left| \begin{array}{cc} (k_0 + i\,K\mathbf{e}\,\vec{\sigma}) & (n_0 - iN\,\mathbf{e}\,\vec{\sigma}) \\ -(n_0 - iN\,\mathbf{e}\,\vec{\sigma}) & (k_0 + iK\,\mathbf{e}\,\vec{\sigma}) \end{array} \right| . \qquad (85)$$

It should be noted that

$$\det G = (k_0^2 + K^2 + n_0^2 + N^2)^2 = +1\,. \qquad (86)$$

This result may be verified by direct calculation. Indeed,

$$G^{+} = \left| \begin{array}{cc} (k_0 - i\,K\mathbf{e}\,\vec{\sigma}) & -(n_0 + iN\,\mathbf{e}\,\vec{\sigma}) \\ (n_0 + iN\,\mathbf{e}\,\vec{\sigma}) & (k_0 - iK\,\mathbf{e}\,\vec{\sigma}) \end{array} \right| ,$$

and further for $GG^{+} = I$ we get (by 2×2 blocks)

$$(GG^{+})_{11} = k_0^2 + K^2 + n_0^2 + N^2 = +1\,, \qquad (GG^{+})_{12} = -2i(n_0 K + k_0 N)\,(\mathbf{e}\,\vec{\sigma}) = 0\,,$$
$$(GG^{+})_{22} = k_0^2 + K^2 + n_0^2 + N^2 = +1\,, \qquad (GG^{+})_{21} = +2i(n_0 K + k_0 N)\,(\mathbf{e}\,\vec{\sigma}) = 0\,.$$

One different way to parameterize (84) can be proposed. Indeed, relations (85) are

$$k_0, \qquad \mathbf{k} = K\mathbf{e}, \qquad n_0, \qquad \mathbf{n} = N\mathbf{e}\,,$$
$$k_0^2(1+\frac{K^2}{k_0^2}) + n_0^2(1+\frac{N^2}{n_0^2}) = +1\,, \qquad \frac{K}{k_0} = -\frac{N}{n_0} \equiv W\,, \tag{87}$$

or

$$k_0, \qquad \mathbf{k} = k_0 W\,\mathbf{e}, \qquad n_0, \qquad \mathbf{n} = -n_0 W\,\mathbf{e}\,,$$
$$(k_0^2+n_0^2)\,(1+W^2) = +1\,,$$
$$K = k_0 W, \qquad N = -n_0 W\,, \qquad 0 \le k_0^2+n_0^2 \le 1\,. \tag{88}$$

Therefore, matrix G can be presented as follows:

$$G = \left| \begin{array}{cc} k_0(1+i\,W\mathbf{e}\,\vec{\sigma}) & n_0(1+iW\,\mathbf{e}\,\vec{\sigma}) \\ -n_0(1+iW\,\mathbf{e}\,\vec{\sigma}) & k_0(1+iW\,\mathbf{e}\,\vec{\sigma}) \end{array} \right|, \tag{89}$$
$$(k_0^2+n_0^2)\,(1+W^2) = +1\,, \qquad \Longrightarrow \qquad W = \pm\sqrt{\frac{1}{k_0^2+n_0^2}-1}\,.$$

Evidently, it suffices to take positive values for W. The form (89) for G shows that the unitary sub-group depends upon four parameters $k_0,\ n_0,\ \mathbf{e}$:

$$0 \le k_0^2+n_0^2 \le 1\,, \qquad \mathbf{e}^{\,2} = 1\,. \tag{90}$$

Let us establish the law of multiplication for four parameters $k_0, n_0, \mathbf{W} = W\mathbf{e}$:

$$G'' = G'G = \left| \begin{array}{cc} k_0'(1+i\mathbf{W}'\,\vec{\sigma}) & n_0'(1+i\mathbf{W}'\,\vec{\sigma}) \\ -n_0'(1+i\mathbf{W}'\,\vec{\sigma}) & k_0'(1+i\mathbf{W}'\,\vec{\sigma}) \end{array} \right| \left| \begin{array}{cc} k_0(1+i\mathbf{W}\,\vec{\sigma}) & n_0(1+i\mathbf{W}\,\vec{\sigma}) \\ -n_0(1+i\mathbf{W}\,\vec{\sigma}) & k_0(1+i\mathbf{W}\,\vec{\sigma}) \end{array} \right| \tag{91}$$

or by 2×2 blocks

$$(11) = \ \ (k_0'k_0 - n_0'n_0)\,(1+i\mathbf{W}'\,\vec{\sigma})(1+i\mathbf{W}\,\vec{\sigma})\,,$$
$$(12) = \ \ (k_0'n_0 + n_0'k_0)\,(1+i\mathbf{W}'\,\vec{\sigma})(1+i\mathbf{W}\,\vec{\sigma})\,,$$
$$(21) = -(k_0'n_0 + n_0'k_0)\,(1+i\mathbf{W}'\,\vec{\sigma})(1+i\mathbf{W}\,\vec{\sigma})\,,$$
$$(22) = \ \ (k_0'k_0 - n_0'n_0)\,(1+i\mathbf{W}'\,\vec{\sigma})(1+i\mathbf{W}\,\vec{\sigma})\,.$$

Because $(11) = (22),\ (12) = -(21)$; further one can consider only two blocks:

$$(11) = \ (k_0'k_0 - n_0'n_0)(1-\ \mathbf{W}'\mathbf{W})\ (\ 1+i\frac{\mathbf{W}'+\ \mathbf{W}-\ \mathbf{W}'\times\mathbf{W}}{1-\ \mathbf{W}'\mathbf{W}}\ \vec{\sigma}\ )\,,$$

$$(12) = \ (k_0'n_0 + n_0'k_0)(1-\ \mathbf{W}'\mathbf{W})\ (\ 1+i\frac{\mathbf{W}'+\ \mathbf{W}-\ \mathbf{W}'\times\mathbf{W}}{1-\ \mathbf{W}'\mathbf{W}}\ \vec{\sigma}\ )\,.$$

So the composition rule should be

$$k_0'' = (k_0'k_0 - n_0'n_0)\,(1-\ \mathbf{W}'\mathbf{W})\,, \qquad n_0'' = (k_0'n_0 + n_0'k_0)\,(1-\ \mathbf{W}'\mathbf{W})\,,$$
$$\mathbf{W}'' = \frac{\mathbf{W}'+\ \mathbf{W}-\ \mathbf{W}'\times\mathbf{W}}{1-\ \mathbf{W}'\mathbf{W}}\,. \tag{92}$$

It remains to prove two the identity $(k_0''^2 + n_0''^2)\,(1+W''^2) = +1$, which reduces to

$$(k_0'k_0 - n_0'n_0)^2 + (k_0'n_0 + n_0'k_0)^2\ \ \left[(1-\ \mathbf{W}'\mathbf{W})^2 + (\mathbf{W}'+\ \mathbf{W}-\ \mathbf{W}'\times\mathbf{W})^2\right] = 1\,. \tag{93}$$

First terms is

$$(k'_0 k_0 - n'_0 n_0)^2 + (k'_0 n_0 + n'_0 k_0)^2 = (k_0^{'2} + n_0^{'2})\ (k_0^2 + n_0^2)\ .$$

Second term is

$$(1 - \ \mathbf{W}'\mathbf{W})^2 + (\mathbf{W}' + \ \mathbf{W} - \ \mathbf{W}' \times \mathbf{W})^2 = (1 + \mathbf{W}^{'2})(1 + \mathbf{W}^2)\ .$$

Therefore, (93) take the forme

$$(k_0^{'2} + n_0^{'2})\ (k_0^2 + n_0^2)(1 + \mathbf{W}^{'2})(1 + \mathbf{W}^2) = 1\ ;$$

which is identity due to equalities:

$$(k_0^{'2} + n_0^{'2})\ (1 + \mathbf{W}^{'2}) = 1\ , \qquad (k_0^2 + n_0^2)(1 + \mathbf{W}^2) = 1\ .$$

It is matter of simple calculation to introduce a curvilinear parameters for such a unitary subgroup:

$$\mathbf{e} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)\ , \tag{94}$$
$$k_0 = \cos\alpha\cos\rho,\ K = \cos\alpha\sin\rho\ , \qquad n_0 = \sin\alpha\cos\rho,\ N = -\sin\alpha\sin\rho\ ,$$

and G looks as follows

$$G = \left| \begin{array}{cc} \Delta & \Sigma \\ -\Sigma & \Delta \end{array} \right| , \qquad \Delta = \left| \begin{array}{cc} \cos\alpha(\cos\rho + i\sin\rho\cos\theta) & i\cos\alpha\sin\rho\sin\theta e^{-i\phi} \\ i\cos\alpha\sin\rho\sin\theta e^{i\phi} & \cos\alpha(\cos\rho - i\sin\rho\cos\theta \end{array} \right| ,$$
$$\Sigma = \left| \begin{array}{cc} \sin\alpha(\cos\rho + i\sin\rho\cos\theta) & +i\sin\alpha\sin\rho\ \sin\theta e^{-i\phi} \\ i\sin\alpha\sin\rho\ \sin\theta e^{i\phi} & \sin\alpha(\cos\rho - i\sin\rho\cos\theta) \end{array} \right| . \tag{95}$$

In conclusion let us list main results:

Parametrization of 4×4 – matrices G of the complex linear group $GL(4, C)$ in terms of four complex vector-parameters $G = G(k, m, n, l)$ is investigated. In the given parametrization, the problem of inverting any 4×4 matrix G is solved. Expression for determinant of any matrix G is found: det $G = F(k, m, n, l)$. Unitarity conditions have been formulated in the form of non-linear cubic algebraic equations including complex conjugation. Two simplest types of solutions have been constructed: 1-parametric Abelian sub-group G_0 of 4×4 unitary matrices; three 2-parametric sub-groups G_1, G_2, G_3; one 4-parametric unitary sub-group. Curvilinear coordinates to cover these sub-groups have been found.

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References

[1] A.A. Bogush, V.M, Red'kov, Vesti National Academy of Sciences of Belarus, Ser fiz.-mat., **N 3** 63-69 (2006).

[2] A.A. Bogush, V.M. Red'kov, On unique parametrization of the linear group $GL(4.C)$ and its subgroups by using the Dirac matrix algebra basis. hep-th/0607054; Submitted to journal: Nonlinear Phenomena in Complex Systems.

[3] F.I. Fedorov, *The Lorentz group.* (Moscow, 1979).