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Level crossing for quantum systems with some degrees of freedom

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Complex systems, such as nuclear, atomic, molecular ones and so on, are characterized by some degrees of freedom, separation of which in strongly interacting systems, as a rule, are not valid. One of the most useful methods in the treatment of the quantum dynamical systems with some degrees of freedom is the adiabatic representation method.

Here we investigate the role played by level crossing for collective motion in the presence of "fast" dynamics of separate particles in the system within the adiabatic representation. The method presented permits one to construct a wide class of potentials and corresponding solutions of the parametric equation in a closed analytical form and, after that, to calculate the matrix elements of the exchange interaction. It was shown that the main features of the exchange interaction determining the "slow" subsystem Hamiltonian essentially depend on the character of the parametric Hamiltonian: namely, it is given on the semi–axis or on the entire axis. As a consequence, the problems of level crossing are different in both cases. It was established that in the case of the parametric problem on the entire axis the induced scalar and vector potentials and the basis functions are not singular at the degeneracy points of the two states, while in the parametric problem on the half-axis, the potential, together with its eigenfunctions and matrix elements of the exchange interaction, are singular at these points. In particular, we have found that in the parametric problem on the entire axis for a special choice of the normalization functions, the potential is transparent and symmetric in the "fast" variable and the exchange interaction between the bound states for two-level systems are equal to zero for all values of the "slow" variables, even at the point of the degeneracy.

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1. Introduction

The adiabatic representation method is elaborated on the basis of a consistent formulation of both connected problems: the parametric problem and the multichannel problem for the system of equations with a covariant derivative. In this approach the Hamiltonian H is decomposed into

$$H = h^s \otimes I + h^f, \tag{1}$$

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where $h^f \equiv h^f(x)$ is the family of Hamiltonians depending parametrically on the slow variables. The searched wave function $\Psi(X)$ ($X = \{x, y\}$) of the total Hamiltonian is given by expansion

$$|\Psi(X) >= \sum_n |n >< n|\Psi >= \sum_n \int \psi_n(x; y) F_n(x) \tag{2}$$

in the $\psi_n(x; y)$ self-adjoint parametric Hamiltonian $h^f(x) \equiv h(x)$

$$\begin{aligned} h(x)|\psi_n(x; y) >= \mathcal{E}_n(x)|\psi_n(x; y) > \\ h(x) = -\Delta_y + V(x, y). \end{aligned} \tag{3}$$

Since the Hamiltonian $h(x)$ is self-adjoint, its eigenfunctions form a complete orthonormal set

$|\psi_n(x; y) >< \psi_n(x; y')| = \delta(y - y')$,

$< \psi_n(x; y) \mid \psi_m(x; y) >= \delta_{nm} \quad \forall x$,

with elements depending on x parametrically. Upon substituting the expansion (2) into the initial Schrödinger equation

$$H\Psi(X) = E\Psi(X) \tag{4}$$

and using the relations of orthonormalization, we arrive at a multichannel system of gauge equations

$$[-(\nabla_x + A(x))^2 + V(x) - P^2]F(x) = 0. \tag{5}$$

Here $A(x)$ and $V(x)$ are the effective vector and scalar potentials, respectively, the matrix elements of which are induced by the basis functions $\psi(x; y)$ of the parametric problem (3)

$$A_{nm}(x) =< \psi_n(x; y)|i\nabla_x|\psi_m(x; y) >, \tag{6}$$

$$\begin{aligned} V_{nm}(x) &= < \psi_n(x; y)|h(x)|\psi_m(x; y) >\equiv \\ &\equiv \mathcal{E}_n(x)\delta_{nm}. \end{aligned} \tag{7}$$

Here, we assume that $h(x)$ is real, limited, and continuous in x. Because, for each x, the eigenfunctions are real valued and orthonormal then the induced couplings $A_{nm} = -A_{mn}$ in (6) are real and antisymmetric in n and m. The matrix elements of (6) of the induced connection A can be computed in terms of the eigenfunctions of parametric equations (3) for the given functional dependence of scattering data $\{S(x, k), M^2(x), \mathcal{E}(x)\}$ on the slow coordinate variable x.

In accordance with the general definition of the inverse scattering problem [1] and [2], the parametric inverse problem [3] consist of the reconstruction of the potential and corresponding solutions from the known scattering data $\{S(x, k), M^2(x), \mathcal{E}(x)\}$ (in the Marchenko approach) or the spectral data $\{\rho(x, k), N^2(x), \mathcal{E}(x)\}$ (Gelfand-Levitan approach) parametrically depending on the coordinate variable x. This dependence reflects the peculiarity of the nonstandard parametric inverse problem. Specifying this dependence and employing the algebraic methods of the inverse scattering problem, we present a wide class of potentials for which one can construct exactly solvable models and, consequently, derive solutions in a closed analytic form. These generalized Bargmann potentials are defined by the rational Jost functions $f \equiv f_+$,

$$f(x; k) = \overset{\circ}{f}(k) \prod \frac{k - i\alpha(x)}{k + i\beta(x)} \tag{8}$$

parametrically depending on the "slow" dynamical variables x through the dependence of spectral parameters on these variables. This situation is, to a certain extent, analogous to the theory

of nonlinear evolution equations. The parametric Jost function (8) has N curves $k = -i\beta_j(x)$, $j = 1, 2, ...N$, of simple poles and N curves of simple zeros $k = i\alpha_j(x)$ defined as functions of the parametric variable x.

For real potentials, the curves $i\alpha_j(x)$ and $-i\beta_j(x)$ must be situated symmetrically with respect to the imaginary axis in the complex k plane. In $\alpha(x)$, there are not only zeros on the imaginary semi-axis corresponding to the bound states $Re\ \kappa_j(x) = 0$, $Im\ \kappa_j(x) > 0$ for each value of x, but, also, zeros in the lower k half-plane with $Im\ \nu_j(x) < 0$ (the number of simple pole curves of $\beta_j(x)$ equals the total number of $\kappa_j(x)$ and $\nu_j(x)$). In this case, the scattering matrix and the spectral function assume the form

$$\mathcal{S}(x;k) = \overset{\circ}{S}(k) \prod \frac{(k+i\alpha(x))(k+i\beta(x))}{(k-i\beta(x))(k-i\alpha(x))},$$
$$\rho(x;k) = \overset{\circ}{\rho}(k) \prod \frac{(k-i\beta(x))(k+i\beta(x))}{(k+i\alpha(x))(k-i\alpha(x))}. \tag{9}$$

For such $\mathcal{S}(x;k)$ and $\rho(x;k)$, the kernels of the integral equations of the parametric inverse problem can be represented as the sums of terms with a factorized dependence on the fast variable y: $Q(x;y,y') = \sum_i^N B_i(x;y)B_i(x;y')$. When the kernel Q is inserted into the base parametric equation of the inverse problem,

$$K(x;y,y') + Q(x;y,y') + \int_{y(0)}^{\infty(y)} K(x;y,y'')Q(x;y'',y')dy' = 0, \tag{10}$$

it is evident that the kernel of the generalized shift $K(x;y,y')$ also becomes degenerate:
$K(x;y,y') = \sum_i^N K_i(x;y)B_i(x;y')$.
As a consequence, the system of integral equations of the inverse problem is reduced to a system of algebraic equations.

Then, the spherically nonsymmetric potential and solutions corresponding to it can be expressed in a closed analytic form in terms of the known solutions and spectral characteristics by using the generalized equations of the parametric inverse problem,

$$V(x,y) = \overset{\circ}{V}(y) \mp 2\frac{d}{dy}K(x;y,y), \tag{11}$$

$$\phi(x;k,y) = \overset{\circ}{\phi}(k,y) + \int_{y(0)}^{\infty(y)} K(x;y,y')\ \overset{\circ}{\phi}(k,y')dy'. \tag{12}$$

Integration limits in (10), (12), and the signs in (11) depend on the particular approach to the inverse problem. The limits from y to ∞ and the minus sign correspond to the Marchenko approach. Limits $[0,y]$ and plus sign represent the Gelfand–Levitan approach.

Within the generalized Marchenko approach [4], the integral kernels $Q(x;y,y')$, dependent

on x as a parameter

$$Q(x;y,y') = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\overset{\circ}{\mathcal{S}}(k)-\mathcal{S}(x;k)]\overset{\circ}{f}(k,y)\overset{\circ}{f}(k,y')dk + \sum_{n}^{m}M_n^2(x)\overset{\circ}{f}(i\kappa_n(x),y)\overset{\circ}{f}(i\kappa_n(x),y') - \sum_{n}^{\overset{\circ}{m}}\overset{\circ}{M}{}_n^2\overset{\circ}{f}(i\overset{\circ}{\kappa}_n,y)\overset{\circ}{f}(i\overset{\circ}{\kappa}_n,y'), \quad (13)$$

are constructed by using two sets of the scattering data. These are the set $\{\mathcal{S}(x;k),\ \mathcal{E}_n(x),\ M_n^2(x)\}$, corresponding to equation (3) for every value of parameter x, and the ordinary scattering data $\{\overset{\circ}{\mathcal{S}}(k),\overset{\circ}{\mathcal{E}}_n,\overset{\circ}{M}{}_n^2\}$, corresponding to (3) with $V(x;y)=0$ and $\overset{\circ}{V}(y)\neq 0$. The functions $\overset{\circ}{f}(k,y)$ are standard Jost solutions with the known potential $\overset{\circ}{V}(y)$. Potentials (11) and Jost solutions (12) are determined from $K(x;y,y')$, with respect to which, the linear integral equation (10) is solved for every fixed x.

2. Exactly solvable models within the parametric inverse problem on the semi-axis

For the parametric inverse problem, radial or on a semi-axis, when $\overset{\circ}{V}(y)=0$ and, correspondingly,
$\overset{\circ}{\mathcal{S}}(k)=1$, the kernel of the basic integral equation (10) in the Marchenko approach,

$$Q(x;(y+y')) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[1-\mathcal{S}(x;k)]\exp[ik(y+y')]\,dk + \sum_{n}^{N}M_n^2(x)\exp[-\kappa_n(x)(y+y')], \quad (14)$$

with the scattering matrix (9), can be rewritten as

$$\begin{aligned} & Q(x;(y+y')) = \\ & -\sum_{n}^{N}Res\ \mathcal{S}(k=i\beta_n(x))\exp[-\beta_n(x)(y+y')] \quad (15)\\ & +\sum_{n}^{N}\{-iRes\ \mathcal{S}(k=i\kappa_n(x))\exp[-\kappa_n(x)(y+y')] \\ & +\ M_n^2(x)\exp[-\kappa_n(x)(y+y')]\}\ . \end{aligned}$$

Following the procedure of constructing phase–equivalent potentials suggested in [5] for the one-dimensional problem and in [3] for the parametric problem, we shall reduce solution of the inverse problem to the successive applications of two procedures. First, one can cancel out the second summation in the right-hand side of (15) if the normalization functions $M_n^2(x)=\overset{\circ}{M}{}_n^2(x)$

are chosen to be equal to $i\ Res\ \mathcal{S}(k)$ at $k = i\kappa_n(x)$,

$$\begin{aligned} \overset{\circ}{M}{}_n^2(x) &= i\ Res\ \mathcal{S}(k)|_{k=i\kappa_n(x)} = \\ &= -\frac{2\kappa_n(x)(\kappa_n(x)+\beta_n(x))}{(\kappa_n(x)-\beta_n(x))} \times \\ &\times \prod_{n'\neq n}^{N} \frac{(\kappa_n(x)+\beta_{n'}(x))(\kappa_n(x)+\kappa_{n'}(x))}{(\kappa_n(x)-\beta_{n'}(x))(\kappa_n(x)-\kappa_{n'}(x))}. \end{aligned} \tag{16}$$

As a result, we obtain a simpler expression for the kernel $Q \equiv \overset{\circ}{Q}$,

$$\overset{\circ}{Q}(x;y,y') = \sum_{n}^{N} A_n(x)\exp[-\beta_n(x)(y+y')], \tag{17}$$

where

$$\begin{aligned} A_n(x) &= \frac{2\beta_n(x)(\beta_n(x)+\kappa_n(x))}{(\beta_n(x)-\kappa_n(x))} \times \\ &\times \prod_{n'\neq n}^{N} \frac{(\beta_n(x)+\kappa_{n'}(x))(\beta_n(x)+\beta_{n'}(x))}{(\beta_n(x)-\beta_{n'}(x))(\beta_n(x)-\kappa_{n'}(x))}. \end{aligned} \tag{18}$$

Inserting the kernel $\overset{\circ}{Q}(x;y,y')$ (17) into the parametric Marchenko equation (10), we obtain

$$\begin{aligned} \overset{\circ}{K}(x;y,y') \ + \ \sum_{n}^{N} A_n(x)\Big\{ e^{-\beta_n(x)y} \ + \\ + \ \int_{y}^{\infty} \overset{\circ}{K}(x;y,y'')e^{-\beta_n(x)y''}dy''\Big\}e^{-\beta_n(x)y'} = 0, \end{aligned}$$

where the expression in braces is the Jost solution

$\overset{\circ}{f}(k = i\beta_n(x), y)$ for the sought potential $\overset{\circ}{V}(x;y)$. Such that $\overset{\circ}{K}(x;y,y')$ has a form similar to that of

$\overset{\circ}{Q}(x;y,y')$ from (17) with a separable dependence on y and y' and with a parametric dependence on x,

$$\overset{\circ}{K}(x;y,y') = -\sum_{n}^{N} A_n(x)\ \overset{\circ}{f}(i\beta_n(x),y)\exp[-\beta_n(x)y']. \tag{19}$$

Substituting this kernel of the generalized shift from the free wave to the Jost solution into the triangular integral equation

$$\overset{\circ}{f}(x;k,y) = \exp(iky) + \int_{y}^{\infty} \overset{\circ}{K}(x;y,y')\exp(iky')dy'$$

we get, at $k = i\beta_n(x)$, a set of equations for $\overset{\circ}{f}(i\beta_n(x),y)$,

$$\overset{\circ}{f}(i\beta_n(x),y) = \sum_{n'}^{N} \exp[-(\beta_{n'}(x)y]P^{-1}_{n'n}(x;y),$$

where $P_{nn'}(x;y)$ is defined as follows:

$$P_{nn'}(x;y) = \delta_{nn'} + A_n(x)\frac{\exp[-(\beta_n(x)+\beta_{n'}(x))y]}{\beta_n(x)+\beta_{n'}(x)}.$$

Then, by substituting (19) into the parametric equations of the inverse problem (11), (12), we obtain

$$\overset{\circ}{V}(x;y) = -2\frac{d^2}{dy^2}\ln\det||P(x;y)||, \tag{20}$$

$$\begin{aligned}&\overset{\circ}{f}_{\pm}(x;k,y) = exp(\pm iky) + \\ &+ \sum_{nn'}^{N} A_n(x)P^{-1}_{nn'}(x;y)\frac{\exp[-(\beta_n(x)+\beta_{n'}(x)\mp ik)y]}{(\beta_n(x)\mp ik)}.\end{aligned} \tag{21}$$

A similar situation is obtained in the Gelfand–Levitan approach, only, there, the normalizing functions $N_n^2(x)$ are expressed through $Res\,[f_+(x;k)f_-(x;k)]^{-1}$ at the points $k = i\kappa_n(x)$:

$$N_n^2(x) = 4i\kappa_n^2(x)[(df_+(x;k)/dk)|_{k=i\kappa_n(x)}f_-(i\kappa_n(x))]^{-1}.$$

The corresponding algebraic formulae for the one-dimensional Bargmann potentials and their solutions [5] can be obtained directly if we set $\kappa_n(x) \equiv \kappa_n$ and $\beta_n(x) \equiv \beta_n$.

At the second stage, by using the corresponding $\overset{\circ}{f}(x;k,y)$ as the initial solutions, we obtain a family of potentials and solutions for arbitrary normalizing functions $M_n^2(x)$ that do not obey the condition (16): $\overset{\circ}{M}{}_n^2(x) < M_n^2(x)$. Since the scattering function $\mathcal{S}(x;k)$ is independent of the choice of the normalization functions $M_n^2(x)$, we have $\mathcal{S}(x;k) = \overset{\circ}{\mathcal{S}}(x;k)$. As a result, the integral term in a generalized expression such as (13) for $Q(x;y,y')$ vanishes. Since, on the other hand, both $V(x;y)$ and $\overset{\circ}{V}(x;y)$ possess the same potential curves (curves of bound states) $\mathcal{E}_n(x) = \overset{\circ}{\mathcal{E}}_n(x)$ but different normalization factors $M_n^2(x)$ and $\overset{\circ}{M}{}_n^2(x)$, respectively, we find that

$$\begin{aligned}&Q(x;y,y') = \\ &= \sum_{n}^{N}(M_n^2(x) - \overset{\circ}{M}{}_n^2(x))\,\overset{\circ}{f}(i\kappa_n(x),y)\,\overset{\circ}{f}(i\kappa_n(x),y').\end{aligned} \tag{22}$$

And similarly, the kernel of the generalized shift $K(x;y,y')$ is written as

$$\begin{aligned}&K(x;y,y') = \\ &= -\sum_{n}^{N}(M_n^2(x) - \overset{\circ}{M}{}_n^2(x))f(i\kappa_n(x),y)\,\overset{\circ}{f}(i\kappa_n(x),y').\end{aligned} \tag{23}$$

Inserting $K(x;y,y')$ and $Q(x;y,y')$ into the basic parametric Marchenko equations (10)–(12), we derive the following relations for the potential and Jost solutions:

$$V(x;y) = \overset{\circ}{V}(x;y) + 2\frac{d^2}{dy^2}\ln\det||P(x;y)||\,, \tag{24}$$

$$f_{\pm}(x;k,y) = \overset{\circ}{f}_{\pm}(x;k,y) - \sum_{nm}^{N}(M_n^2(x) - \overset{\circ}{M}{}_n^2(x)) \times$$
$$\times \overset{\circ}{f}(i\kappa_n(x),y)P_{nm}^{-1}(x;y)\int_y^{\infty}\overset{\circ}{f}(i\kappa_m(x),y')\overset{\circ}{f}_{\pm}(k,y')dy'.$$

The explicit dependence on the fast variables is defined by the Jost solutions (21) determined at $k = i\kappa_n(x)$, i.e., on the level-energy curves depending on the parametric variable x. Here we employed the notation

$$P_{nm}(x;y) = \delta_{nm} + (M_n^2(x) - \overset{\circ}{M}{}_n^2(x)) \times$$
$$\times \int_y^{\infty}\overset{\circ}{f}(i\kappa_n(x),y')\overset{\circ}{f}(i\kappa_m(x),y')dy'.$$

Since $\mathcal{S}(x;k)$, corresponding to the two-dimensional potentials $V(x;y)$ (24), is independent of the normalizations $M_n^2(x)$, $\mathcal{S}(x;k) = \overset{\circ}{\mathcal{S}}(x;k)$, the formula (24) represents a parametric family of potentials depending on N parametric functions $M_n^2(x)$. A change of the normalization functions $M_n^2(x)$ of the parametric eigenstates leads to a change in the potentials, the Jost, regular, and basis solutions of the parametric Hamiltonian, and the matrix elements of the exchange interaction, and, therefore, strongly influences the behavior of the quantum systems.

2.1. Bargmann potentials with two potential curves

Let us now present the case with two potential curves in the problem on the semi-axis. The Jost function (8) can be written in the form

$$f(x;k) = \frac{(k - i\kappa_1(x))(k - i\kappa_2(x))}{(k + i\beta_1(x))(k + i\beta_2(x))};$$
$$Im\ \{\kappa_j(x)\} = Im\ \{\beta_j(x)\} = 0, \quad j = 1,2 \tag{25}$$

and

$$\mathcal{S}(x;k) = \tag{26}$$
$$= \frac{(k + i\kappa_1(x))(k + i\beta_1(x))}{(k - i\beta_1(x))(k - i\kappa_1(x))}\frac{(k + i\kappa_2(x))(k + i\beta_2(x))}{(k - i\beta_2(x))(k - i\kappa_2(x))}.$$

Two pole curves correspond to the zero curves $k = i\kappa_j(x)$ of the parametric Jost function $f_+(x;k))$, the other pole curves correspond to the pole trajectories $k = i\beta_j(x)$ of the parametric Jost function $f_-(x;k)$.

We have, necessarily, $\beta_j(x) > 0$ (for the Jost function $f_+(x;k)$ to be analytic in the upper half-plane k for all x). When $\kappa_j(x) > 0$, we have the bound state curves (the potential curves of bound states $\mathcal{E}_j(x) = -\kappa_j^2(x)$); when $\kappa_j(x) < 0$, we do not have any bound states. In principle, the functions $\alpha(x)$ in (8), (9) can change from negative values $\alpha(x) = -\nu(x)$ (anti-bound state trajectory) to positive values $\alpha(x) = \kappa(x)$. If $\alpha_j(x) = -\nu_j(x) < 0 \ \ \forall x$, the potential is not deep and wide enough to produce bound states and the potential corresponds to $\mathcal{S}(x,k)$ with pole curves at $k = i\beta_j(x)$. Here we assume that $\kappa_j(x) > 0$, to provide for the existence of two bound state curves. The ordering of the potential curves $\mathcal{E}_i < \mathcal{E}_{i+1}$ is assumed. To simplify the investigation of the problem, we take the normalizations of the bound state wave functions in the form (16)

$$M_1^2(x) = \tag{27}$$

$$= -\frac{2\kappa_1(x)(\kappa_1(x)+\beta_1(x))}{(\kappa_1(x)-\beta_1(x))}\frac{(\kappa_1(x)+\beta_2(x))(\kappa_1(x)+\kappa_2(x))}{(\kappa_1(x)-\beta_2(x))(\kappa_1(x)-\kappa_2(x))},$$

$$M_2^2(x) =$$

$$= -\frac{2\kappa_2(x)(\kappa_2(x)+\beta_2(x))}{(\kappa_2(x)-\beta_2(x))}\frac{(\kappa_2(x)+\beta_1(x))(\kappa_2(x)+\kappa_1(x))}{(\kappa_2(x)-\beta_1(x))(\kappa_2(x)-\kappa_1(x))}.$$

Thus, the potential $V(x,y)$ is determined only by the spectral data $\kappa_j(x)$ and $\beta_j(x)$, $j=1,2$, and corresponds to one of the family of potentials characterized by the same energy levels $\mathcal{E}_j(x)$ and the same parametric $\mathcal{S}(x;k)$ (27) with four pole curves. From the relations (20), (21), we obtain the two–dimensional potential $V(x,y)$ and the corresponding normalized wave functions

$$\psi_j(x,y) = M_j(x) f(i\kappa_j(x), y), \quad j=1,2$$

of the self-energy curves $\mathcal{E}_{1,2}(x)$. The matrix element $A_{12}(x) = \int \psi_1(x;y)\partial_x\psi_2(x;y)dy$ of the induced connection (6) is computed in terms of the analytic functions $\psi_{1,2}(x;y)$.

From the normalizations $M_j^2(x)$ being positive definite, the conditions $\beta_2(x) \geq \kappa_2(x)$ and $\beta_1(x) \geq \kappa_1(x) \geq \beta_2(x)$ follow. This means that $\mathcal{E}_2(x) \geq \beta_2^2(x) \geq \mathcal{E}_1(x)$. If the levels $\mathcal{E}_1(x)$ and $\mathcal{E}_2(x)$ move towards each other, one or both of them would be equal to $\beta_2^2(x)$ at any point of $x = x'$. It can be easily seen from the relations (27) that the corresponding normalizing function $M_1^2(x)$ or $M_2^2(x)$ becomes singular when $\kappa_1(x') = \beta_2(x')$ or $\kappa_2(x') = \beta_2(x')$. If $\kappa_1(x') = \beta_2(x') = \kappa_2(x')$, both normalization functions $M_1^2(x)$ and $M_2^2(x)$ have double poles at the point of the degeneracy $x = x'$. It can be seen from the relations (19) and (20) that the potential $V(x;y)$ with the pertinent normalized functions $\psi_{1,2}(x,y)$ has a double pole at this point $x = x'$ somewhere on the positive y-axis, and the matrix elements of the induced vector potentials (6), determined by $\psi_{1,2}(x,y)$, are singular, too. One can see from (16) that the same investigations with singular behaviors of normalizing functions are valid for potentials with an arbitrary number of levels when two levels move closer together. Remind of the fist investigations of normalizations in [6] at avoided level crossings.

3. Transparent Potentials

The one-dimensional inverse problem on the entire axis $-\infty < y < \infty$ with the zero-th reflection coefficient, $\mathcal{S}^{ref}(k) = 0$, describes transparent (reflectionless) potentials along the variable y. If the reflection function $\mathcal{S}^{ref}(x;k)$ is chosen to be equal to zero at all energies and at all values of the parametric variable x, then the integral in the relation for $Q(x;y,y')$ vanishes and only the sum over the bound states remains. The transmission coefficient $\mathcal{S}^{tr}$, with an absolute value equal to unity, is a rational function,

$$\mathcal{S}^{tr}(x;k) = \prod \frac{k+i\kappa(x)}{k-i\kappa(x)}, \tag{28}$$

depending on the parametric variable x.

The relations for the potentials and solutions can be expressed in terms of normalized eigenfunctions and represented in a most symmetric and convenient form. Following [7], introduce the function

$$\lambda_n(x;y) = \gamma_n(x)\exp(-\kappa_n(x)y).$$

Then the formula for $K(x;y,y')$ can be written as

$$\begin{aligned} K(x;y,y') &= -\sum_n^N \gamma_n(x)\psi_n(x;y)\exp(-\kappa_n(x)y') \\ &= -\sum_n^N \psi_n(x;y)\lambda_n(x;y'). \end{aligned} \tag{29}$$

For the normalized eigenfunctions $\psi_n(x;y)$ from (12), we obtain

$$\psi_n(x;y) = \sum_j^N \lambda_j(x;y) A_{jn}^{-1}(x;y), \tag{30}$$

with the matrix $A_{jn}(x;y)$ given by

$$A_{jn}(x;y) = \delta_{jn} + \frac{\lambda_j(x;y)\lambda_n(x;y)}{\kappa_n(x)+\kappa_j(x)}. \tag{31}$$

Finally, the kernel $K(x;y,y')$ and the potential can be represented as

$$\begin{aligned} K(x;y,y') &= -\sum_n^N \sum_j^N \lambda_j(x;y) A_{jn}^{-1}(x;y) \lambda_n(x;y'), \\ V(x;y) &= -4\sum_n^m \kappa_n(x)\psi_n^2(x;y). \end{aligned} \tag{32}$$

Recall that these relations are obtained for the specific case of the zero reflection function $\mathcal{S}^{ref}(x;k) = 0 \ \ \forall x$.

Exactly solvable models with time-dependent symmetric in y potentials.

Note that the symmetric transparent potentials for each fixed value of x and the pertinent wave functions are completely defined by the energy levels, since in this case the normalizations of the bound state functions are expressed as

$$\gamma_n^2(x) = 2\kappa_n(x) \prod_{m\neq n} \left| \frac{\kappa_m(x)+\kappa_n(x)}{\kappa_m(x)-\kappa_n(x)} \right|. \tag{33}$$

Consider the simple example of two-dimensional exactly solvable models for two-level systems with symmetric in y potentials. Taking the equation (30) for the normalized eigenfunctions $\psi_1(x;y)$ and $\psi_2(x;y)$ and carrying out some simplifications, we obtain

$$\begin{aligned} &\psi_1(x;y) = \\ &\frac{\sqrt{2\kappa_1(x)b(x)}\cosh(\kappa_1(x)y)}{\cosh[(\kappa_1(x)+\kappa_2(x))y] + b(x)\cosh[(\kappa_1(x)-\kappa_2(x))y]}, \\ &\psi_2(x;y) = \\ &= \frac{\sqrt{2\kappa_2(x)b(x)}\sinh(\kappa_2(x)y)}{\cosh[(\kappa_1(x)+\kappa_2(x))y] + b(x)\cosh[(\kappa_1(x)-\kappa_2(x))y]} \end{aligned} \tag{34}$$

with $b(x) = |(\kappa_1(x)+\kappa_2(x))/(\kappa_1(x)-\kappa_2(x))|$. Taking an account of (34) for $V(x;y)$ in (32), we get the potential in an explicit form convenient for analysis:

$$V(x;y) = -8b(x) \times \tag{35}$$

$$\frac{\kappa_1^2(x)\cosh^2(\kappa_1(x)y) + \kappa_2^2(x)\sinh^2(\kappa_2(x)y)}{\{\cosh[(\kappa_1(x)+\kappa_2(x))y] + b(x)\cosh[(\kappa_1(x)-\kappa_2(x))y]\}^2}.$$

Obviously, the potential $V(x;y)$ is symmetric in y for each value of the parametric variable x. It is also easily seen from (34) that the eigenfunction of the ground state $\psi_1(x;y)$ is symmetric and $\psi_2(x;y)$ is antisymmetric in y for each fixed value of x, as it is required for the problem on the entire axis $-\infty < y < \infty$ with a potential $V(x;y)$ symmetric in y for each fixed x.

As a limiting case, we can consider $\kappa_2(x) \to \kappa_1(x)$. It can be seen from (36) and (34) that if $\kappa_2(x) = \kappa_1(x)$ at any point of $x = x'$, the limiting values of the potential and the wave functions are equal to zero, $V(x';y) = \psi_1(x';y) = \psi_2(x';y) = 0 \ \forall y$, while in the above considered problem on the half axis $0 \le y < \infty$, the potential, together with its eigenfunctions, are singular at $x = x'$.

Matrix elements of the exchange interaction $A_{12}(x)$, induced by basis functions of the parametric instantaneous Hamiltonian, can be written in the form

$$\begin{aligned} A_{12}(x) &= 2b(x)\sqrt{\kappa_1(x)\kappa_2(x)} \times \\ \times\Big[\int_{-\infty}^{\infty} &\frac{y(\partial\kappa_2(x)/\partial_x)\cosh(\kappa_1(x)y)\cosh(\kappa_2(x)y)}{G^2(x;y)} \\ -&\frac{\cosh(\kappa_1(x)y)\sinh(\kappa_2(x)y)(\partial G(x;y)/\partial_x)}{G^3(x;y)}dy\Big] + \\ +C(x)\Big[\int_{-\infty}^{\infty} &\frac{\cosh(\kappa_1(x)y)\sinh(\kappa_2(x)y)}{G^2(x;y)}dy\Big] \end{aligned} \tag{36}$$

with $G(x;y) = \cosh[(\kappa_1(x)+\kappa_2(x))y] + b(x)\cosh[(\kappa_1(x)-\kappa_2(x))y]$ and $C(x) = 2\sqrt{b(x)\kappa_1(x)} \times \partial(\sqrt{b(x)\kappa_2(x)})/\partial_x$. It can be easily shown that these integrals vanish. Indeed, $G(x;y)$ and $\partial_x G(x;y)$ are even functions in y and integrands are odd functions of y, i.e., $A_{12}(x) = 0 \ \forall x$. This means that there is no coupling between the eigen-states $\psi_1(x;y)$ and $\psi_2(x;y)$ for any point of the variable x for the transparent symmetric in y potentials. The transition amplitude between the bound state functions $\psi_1(x;y)$ and $\psi_2(x;y)$ is equal to zero because it is defined by the zero matrix elements $A_{12}(x) = 0 \ \ \forall x$.

The choice of normalizing functions $\gamma_n^2(x)$ of the energy-level states, which do not obey the condition (33), leads to loss of symmetry in y of potentials, and to another behavior of the parametric basis functions. As a result, the matrix elements of the exchange interaction $A_{12}(x) \neq 0$, that is the coupling between neighboring states takes place and the transitions take place, as well. As a consequence of our analysis, we can conclude that the choice of normalization functions of the parametric Hamiltonian eigen-states strongly influences the behavior of the quantum systems.

4. Conclusions

The method presented permits one to construct a wide class of potentials and corresponding solutions of the parametric equation (3) in a closed analytical form and, after that, to calculate the matrix elements of the exchange interaction. The first procedure is an algebraic one, but the second step is, in general, numerical. Therefore, the method is semi-analytical. This approach allows one to investigate the influence of the parametric spectral data on the behavior of the potentials, of the basis functions of the parametric Hamiltonian, and of the matrix elements of the exchange interaction. It was shown that the main features of the exchange interaction determining the slow subsystem Hamiltonian essentially depend on the character of the parametric (fast) Hamiltonian: namely, the fast subsystem Hamiltonian is given on the semi–axis $0 \le y < \infty$ or on the entire axis $-\infty < y < \infty$. As a consequence, the problems of level crossing are different in both cases. The matrix elements of the coupling $A_{nm}(x)$ have no singularities at the degeneracy points of two-levels if the consideration is made within the parametric problem on the entire line and $A_{nm}(x)$ are singular at the degeneracy points if the

consideration is made within the parametric problem on the half-axis. It was established that in the case of the parametric problem on the entire axis the induced scalar and vector potentials and the basis functions are not singular at the degeneracy points of the two states, while in the parametric problem on the half-axis, the potential, together with its eigenfunctions and matrix elements of the exchange interaction, are singular at these points. We studied the effect of the normalizing functions on the properties of the quantum systems. In particular, we have found that in the parametric problem on the entire axis for a special choice of the normalization functions (33), the potential $V(x;y)$ is transparent and symmetric in the "fast" variable y and the exchange interaction between the bound states for two-level systems are equal to zero for all values of the "slow" variables, even at the point of the degeneracy. Our approach can be recommended for the investigation of the Landau–Zener transitions and level crossing problems.

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