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Equations of motion for quadrupolar particle in General Relativistic Theory

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In the framework of General Relativity, the equations of orbital and spin motion for a particle with quadrupole moments of mass and electric charge in electromagnetic and gravitational fields are derived. The equations can be used to take account of perturbances caused by the influence of quadrupole moments on the movement of relativistic astrophysical objects.

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Relativistic dynamics of particles with the moments is of interest for a wide range of problems – from movement of high-energy particle through the matter [1, 2] up to the evolution of relativistic astrophysical objects [3]. The methods of derivation of dynamic equations for particles with the moments were widely discussed [4-13]. In the present report, the dynamics of model of a quadrupole particle interacting with electromagnetic and gravitational fields is developed on the basis of the classical general relativistic variational formalism [7, 8].

With the use of the results [7, 8], for derivation of the closed dynamic equations it is sufficient to determine the Lagrange function; this allows to avoid some difficulties inherent in other methods. We shall obtain the Lagrange function for relativistic quadrupole particles as follows. First of all, we shall write the Lagrange function for a system of particles with mass m_a and electric charges q_a in nonrelativistic approach as

$$L = L_{kin} + L_{int}^{gr} + L_{int}^{em}, \tag{1}$$

where

$$L_{kin} = \frac{1}{2} m\, v^2 + \frac{1}{2}\sum_a m_a \,(d\vec{r}_a/dt)^2, \quad L_{int}^{gr} = -m\Phi + \sum_{N=2}^{\infty} L_N^{gr},$$

$$L_N^{gr} = m_{k_1 k_2 \ldots k_{N-1}} g_{k_1,k_2,\ldots k_N}, \quad L_{int}^{em} = q\left(\frac{\vec{v}}{c}\vec{A} - \varphi\right) + \sum_{N=1}^{\infty} L_N^{em},$$

$$L_N^{em} = \mu_{k_1 k_2 \ldots k_{N-1}} H_{k_1,k_2,\ldots k_{N-1}} + q_{k_1 k_2 \ldots k_N}(E_{k_1,k_2,\ldots k_N} + \varepsilon_{k_1 mn}\frac{v_m}{c} H_{n,k_2,\ldots k_N}),$$

$$m = \textstyle\sum_a m_a, \quad q = \sum_a q_a, \quad \vec{r}_a = \vec{x}_a - \vec{x}, \quad \vec{x} = \displaystyle\sum_a \frac{m_a}{m}\vec{x}_a, \quad \vec{v} = \frac{d\vec{x}}{dt},$$

$$m_{k_1 \ldots k_N} = \frac{1}{N!}\sum_a m_a \underset{a}{r}_{k_1} \ldots \underset{a}{r}_{k_N}, \quad q_{k_1 \ldots k_N} = \frac{1}{N!}\sum_a q_a \underset{a}{r}_{k_1} \ldots \underset{a}{r}_{k_N},$$

$$\mu_{k_1 k_2 \ldots k_{N-1}} = \frac{N-1}{N!\, c}\, \varepsilon_{m k_1 k_2} \sum_a q_a \underset{a}{r}_{k_2} \ldots \underset{a}{r}_{k_N} \frac{d}{dt} \underset{a}{r}_m \quad (i,j,k\ldots = 1,2,3) \quad .$$

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Here, the fields $\vec{g} = -\vec{\nabla}\Phi$ (Newton gravitational), $\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ (electric), $\vec{H} = \mathrm{rot}\,\vec{A}$ (magnetic) and derivatives with respect to coordinates (designated by a comma after an index) are calculated at the point $\vec{x}$.

If the system is a rigid "compound" particle, whose orientation is determined by the orts $\vec{e}_k$, we have: $q^k \underset{s}{e}{}_k = \tilde{q}_s = \mathrm{const}$, $q^{ik} \underset{r}{e}{}_i \underset{s}{e}{}_k = \tilde{q}_{rs} = \mathrm{const}$, $m^{ik} \underset{r}{e}{}_i \underset{s}{e}{}_k = \tilde{m}_{rs} = \mathrm{const}$, and in the quadrupole approximation the Lagrange function (1) takes the form

$$\tilde{L} = \frac{mv^2}{2} + \frac{I_{ik}}{2}\omega_i\omega_k - m\Phi + m_{ik}g_{i,k} + q\left(\frac{\vec{v}}{c}\vec{A} - \varphi\right) +$$

$$+q_i\left(E_i + \varepsilon_{ijn}\frac{v_j}{c}H_n\right) + \mu_k H_k + q_{ik}\left(E_{i,k} + \varepsilon_{ijn}\frac{v_j}{c}H_{n,k}\right), \tag{2}$$

where

$$\vec{\omega} = \frac{1}{2}\left[\vec{e}_k \times \frac{d}{dt}\vec{e}_k\right], \quad \mu_i = \frac{1}{2c}\left(q_{nn}\delta_{ik} - q_{ik}\right)\omega_k, \quad I_{ik} = 2\left(m_{jj}\delta_{ik} - m_{ik}\right).$$

Now let us introduce the following general 4-invariant expression:

$$L = -m_0c^2 + \frac{1}{2}I^{\alpha\beta}\omega_\alpha\omega_\beta + m^{\alpha\beta}R_{\mu\alpha\beta\nu}u^\mu u^\nu +$$

$$+\frac{1}{c}qA_\mu u^\mu + \frac{1}{c}q^\alpha F_{\alpha\beta}u^\beta + \frac{1}{c}\mu_\alpha \overset{*}{F}{}^{\alpha\beta} u_\beta + \frac{1}{c}q^{\mu\nu}u^\sigma\nabla_\mu F_{\nu\sigma}, \tag{3}$$

where

$$\overset{*}{F}{}^{\alpha\beta} = \frac{1}{2}\eta^{\alpha\beta\mu\nu}F_{\mu\nu}, \quad \eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\;\varepsilon_{\alpha\beta\gamma\delta}\;\left(\varepsilon_{0123} = -1\right),$$

$$g = \det\left\|g_{\alpha\beta}\right\|, \quad \omega^\alpha = \frac{1}{2c}\eta^{\alpha\beta\gamma\delta}u_\delta \underset{k}{e}{}_\beta \frac{d}{d\tau}\underset{k}{e}{}_\gamma, \quad m_{\alpha\beta} = \tilde{m}_{ik}\underset{i}{e}{}_\alpha \underset{k}{e}{}_\beta,$$

$$I_{\alpha\beta} = \tilde{I}_{ik}\underset{i}{e}{}_\alpha \underset{k}{e}{}_\beta, \quad q_\alpha = \tilde{q}_k \underset{k}{e}{}_\alpha, \quad \mu_\alpha = \frac{1}{c}\left(\tilde{q}_{nn}\delta_{ik} - \tilde{q}_{ik}\right)\underset{k}{e}{}_\alpha \underset{k}{e}{}_\beta\,\omega^\beta.$$

It is easy to verify that $Ld\tau \to (-m_0c^2 + \tilde{L})dt$ (with $\tilde{L}$ from (2)) in the nonrelativistic limit, if $\underset{k}{e}{}^\alpha = (0,\ \vec{e}_k)$ in the particle rest system. Hence, the Lagrange function (3) gives the model of a particle with the moments up to quadrupole ones that interacts with electromagnetic and gravitational fields in GR. The constants $\tilde{q}_i$, $\tilde{q}_{ik}$ and $\tilde{m}_{ik}$ in the expression (3) play roles of the intrinsic characteristics of a particle.

Substituting the Lagrange function (3) into the formulas from [7, 8]:

$$P_\alpha = \frac{\partial L}{\partial u_\alpha} + \left(\frac{\partial L}{\partial u^\lambda}u^\lambda + \frac{\partial L}{\partial \omega^\lambda}\omega^\lambda - L\right)u^\alpha + \frac{\partial L}{\partial \underset{k}{e}{}^\lambda}u^\lambda \underset{k}{e}{}_\alpha + \left[\dot{u} \wedge \frac{\partial L}{\partial \omega}\right]_\alpha + \frac{\partial L}{\partial \omega^\lambda}u^\lambda\omega_\alpha,$$

$$\sigma_\alpha = \frac{\partial L}{\partial \omega^\lambda}n^\lambda_\alpha, \quad F_\alpha = \frac{\partial L}{\partial \Psi}\nabla_\alpha\Psi + \frac{1}{2}\sigma^{\mu\nu}R_{\mu\nu\lambda\alpha}u^\lambda,$$

$$K_\alpha = \left[\underset{k}{e} \wedge \frac{\partial L}{\partial \underset{k}{e}}\right]_\alpha + \left[\omega \wedge \frac{\partial L}{\partial \omega}\right]_\alpha + \frac{\partial L}{\partial \omega^\lambda}\dot{u}^\lambda u_\alpha,$$

where Ψ represents the external fields, one can obtain the dynamic variables for considered model:
– the generalized 4-spin

$$\sigma^{\alpha} = \sigma^{\alpha}_{(m)} + \sigma^{\alpha}_{(q)} = I^{\alpha\beta}\omega_{\beta} + \frac{1}{c}\left(q^{\lambda}_{\lambda}n^{\alpha}_{\beta} - q^{\alpha}_{\beta}\right)H^{\beta}, \tag{4}$$

– the generalized 4-torque

$$K^{\alpha} = \frac{2}{c}\eta^{\nu\lambda}m^{\beta}_{\nu}R_{\sigma\rho\beta\gamma}u^{\sigma}u^{\gamma}u_{\lambda} + q_{\beta}E^{\alpha\beta} +$$

$$+\mu_{\beta}H^{\alpha\beta} + \frac{2}{c}\eta^{\alpha\lambda\beta(\rho}q^{\nu)}_{\beta}u_{\lambda}u^{\sigma}\nabla_{\nu}F_{\rho\sigma} + \frac{1}{c^{2}}\sigma^{\beta}\dot{u}_{\beta}u^{\alpha}, \tag{5}$$

– the generalized 4-momentum

$$P_{\alpha} = m^{*}u_{\alpha} + 2n^{\lambda}_{\alpha}m^{\rho\sigma}R_{\lambda\rho\sigma\nu}u^{\nu} + \frac{1}{c}qA_{\sigma}n^{\sigma}_{\alpha} +$$

$$+\frac{1}{c}q^{\beta}H_{\beta\alpha} + \frac{1}{c}n^{\lambda}_{\alpha}q^{\rho\sigma}\nabla_{\sigma}F_{\rho\lambda} + \frac{1}{c^{2}}\sigma_{\alpha\beta}\dot{u}^{\beta}, \tag{6}$$

where

$$m^{*} = m_{0} + \tfrac{1}{2c^{2}}I^{\rho\sigma}\omega_{\rho}\omega_{\sigma} + \tfrac{\varepsilon_{int}}{c^{2}},$$

$$\varepsilon_{int} = -m^{\rho\sigma}R_{\lambda\rho\sigma\nu}u^{\lambda}u^{\nu} - \frac{q}{c}A_{\lambda}u^{\lambda} - q^{\lambda}E_{\lambda} - \frac{1}{c}q^{\mu\nu}u^{\sigma}\nabla_{\mu}F_{\nu\sigma},$$

– the generalized 4-force

$$F_{\alpha} = \frac{1}{2}\sigma_{\mu\nu}R^{\mu\nu}{}_{\lambda\alpha}u^{\lambda} + m^{\rho\sigma}u^{\gamma}u^{\nu}\nabla_{\alpha}R_{\gamma\rho\sigma\nu} + \frac{q}{c}u^{\lambda}\nabla_{\alpha}A_{\lambda} +$$

$$+\frac{q^{\lambda}}{c}u^{\beta}\nabla_{\alpha}F_{\lambda\beta} + \frac{\mu^{\lambda}}{c}u^{\beta}\,\nabla_{\alpha}\,\overset{*}{F}_{\lambda\beta} + \frac{q^{\sigma\lambda}}{c}u^{\beta}\nabla_{\alpha}\nabla_{\lambda}F_{\sigma\beta}. \tag{7}$$

We use designations:

$$[a\wedge b]^{\alpha} = \eta^{\alpha\beta\gamma\lambda}a_{\beta}b_{\gamma}u_{\lambda},\; n^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \tfrac{1}{c^{2}}u^{\mu}u_{\nu},$$

$$\sigma_{\mu\nu} = \frac{1}{c}\eta_{\mu\nu\lambda\rho}\sigma^{\lambda}u^{\rho}, \quad \omega_{\mu\nu} = \frac{1}{c}\eta_{\mu\nu\lambda\rho}\omega^{\lambda}u^{\rho}, \quad E_{\mu\nu} = n^{\alpha}_{\mu}n^{\beta}_{\nu}\,\overset{*}{F}_{\beta\alpha},$$

$$E^{\mu} = \frac{1}{c}F^{\mu\nu}u_{\nu}, \quad H_{\mu\nu} = n^{\alpha}_{\mu}n^{\beta}_{\nu}F_{\alpha\beta}, \quad H^{\mu} = \frac{1}{c}\overset{*}{F}{}^{\mu\nu}u_{\nu}.$$

Dynamic equations and cinematic constrains [7, 8]

$$\frac{DP_{\mu}}{d\tau} = F_{\mu}, \quad \frac{D\sigma_{\mu}}{d\tau} = K_{\mu}, \tag{8}$$

$$u^{\alpha}(\tau)u_{\alpha}(\tau) = -1, \quad \underset{i}{e}^{\alpha}(\tau)\,\underset{k}{e}_{\alpha}(\tau) = \delta_{ik}, \quad u_{\alpha}(\tau)\;\underset{i}{e}^{\alpha}(\tau) = 0, \tag{9}$$

form the closed system of equations for quadrupole particle in GR. A number of the independent equations in (8) and (9) coincides with a number of unknown functions $x^{\alpha}(\tau)$ and $\underset{k}{e}^{\alpha}(\tau)$. Really, we have 18 equations (8) and (9) for 16 functions $x^\alpha(\tau)$ and $\underset{k}{e}^{\alpha}(\tau)$, but due to (9) two identities are valid: $(\underset{\alpha}{\dot{\sigma}} - K_\alpha)u^\alpha = 0$ and $(\underset{\alpha}{\dot{P}} - F_\alpha)u^\alpha + (\underset{\alpha}{\dot{\sigma}} - K_\alpha)\omega^\alpha = 0$.

Let us note that the tensors $m^{\alpha\beta}$ and $q^{\alpha\beta}$ are connected with standard quadrupole moments of mass $\tilde{Q}_{ik}^{(m)}$ and charge $\tilde{Q}_{ik}^{(q)}$by the relations

$$m^{\alpha\beta} = \frac{1}{3}\tilde{m}_{kk}n^{\alpha\beta} + \frac{1}{6}\tilde{Q}_{ik}^{(m)} \underset{i}{e}^{\alpha} \underset{k}{e}^{\beta}, \quad q^{\alpha\beta} = \frac{1}{3}\tilde{q}_{kk}n^{\alpha\beta} + \frac{1}{6}\tilde{Q}_{ik}^{(q)} \underset{i}{e}^{\alpha} \underset{k}{e}^{\beta}.$$

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