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Spherically-symmetric static space-times with minimally coupled scalar field

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In the paper a general "inverse" method for Einstein equation is developed. It is based on algebraic decomposition of the Einstein tensor. The general family of spherically-symmetric static space-times, filled by a static scalar field, was found by this technique. It is shown that in the case of minimal coupling such space-times is uniquely determined by one characteristic function.

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1. Introduction

Constantly much attention is drown to the problems of algebraic properties of the Riemann tensor and its parts that is invariant under the full Lorentz group: tensors of Weil and Ricci [1, §3.5]. Paradoxically it is not so widely known that analysis of algebraic structure and decomposition of Ricci (or Einstein) tensor drives to complete exhaustive determination of members of some space-time classes [2, 3].

This paper is devoted to demonstration of the technique to account Einstein equations by their algebraic properties. We obtain the general solution, depending on an arbitrary function, for spherically-symmetric static (SSS) space-times, filled by a scalar field with arbitrary physically meaningful Lagrangian.

2. Method of investigation

As it follows from the Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1}$$

the algebraic types of the energy-momentum tensor (EMT) $T_{\mu\nu}$ and the Einstein tensor (ET) $G_{\mu\nu}$ must coincide, so we consequently find the structure of left and right parts of this tensor equation for the concerned class of space-times and match it.

2.1. General form of scalar field EMT

We make use of metric EMT, which can be obtained from Lagrangian $\mathcal{L}$ [4, § 94] by variational derivation with respect to $g_{\mu\nu}$. This choice is motivated by manifest symmetry of the

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obtained tensor, while all other techniques of EMT construction need some additional tricks for the tensor symmetrization.

In the case of a minimally coupled scalar field (as well as for Yang-Mills fields), when there is no explicit dependence of $\mathcal{L}$ on $\Gamma^{\mu}_{\nu\sigma}$, metric EMT reduces to:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial \sqrt{-g}\mathcal{L}}{\partial g^{\mu\nu}} = 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \mathcal{L} g_{\mu\nu} \tag{2}$$

Lagrangian density $\mathcal{L}$ is a scalar under the coordinate transformations, so it must be the function of coordinate scalars. Furthermore, most physically relevant Lagrangians don't incorporate multiple derivatives of the field variables (specific case of GR is not an exception because second order derivatives forms a divergence, that result in disappearing of third derivatives from Einstein equations). So we have very small set of scalars on which Lagrangian can depend.

For real scalar field $\phi(x)$ the only coordinate-invariant scalars of the type considered are ϕ^2 and $g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} = g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}$, for charged one — $\phi^*\phi$ and $g^{\mu\nu}\phi^*_{,\mu}\phi_{,\nu} = g^{\mu\nu}\phi^*_{;\mu}\phi_{;\nu}$ (as usual semicolon denotes covariant and comma denotes usual derivative).

Therefore we find the following general form of real scalar field Lagrangian

$$\mathcal{L} = \mathcal{L}(\phi^2, g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}), \tag{3}$$

and of charged scalar field one

$$\mathcal{L} = \mathcal{L}(\phi^*\phi, g^{\mu\nu}\phi^*_{,\mu}\phi_{,\nu}). \tag{4}$$

These forms include both traditional Lagrangian $\mathcal{L}_0 = \frac{1}{2}(g^{\mu\nu}\phi^*_{,\mu}\phi_{,\nu} - m^2\phi^*\phi)$ and self-action case $\mathcal{L} = \mathcal{L}_0 + V(\phi)$, as well as Lagrangian with modified dynamic term $\mathcal{L} = \frac{1}{2}(f(\phi)g^{\mu\nu}\phi^*_{,\mu}\phi_{,\nu} - m^2\phi^*\phi)$, which is considering now as candidates for dark matter and dark energy [5].

From above the metric EMT of charged scalar field is:

$$T_{\mu\nu} = 2\mathcal{L}'(\phi^*_{,\mu}\phi_{,\nu}) - g_{\mu\nu}\mathcal{L}, \tag{5}$$

where $\mathcal{L}' = \partial\mathcal{L}/\partial(g^{\mu\nu}\phi^*_{,\mu}\phi_{,\nu})$. The charged scalar field being in fact the combination of two independent (by coordinate transformation) real components $\phi_1 = \mathrm{Re}\ \phi$ and $\phi_2 = \mathrm{Im}\ \phi$, in this case the EMT can be represented as

$$T_{\mu\nu} = 2\mathcal{L}'(\phi_{1,\mu}\phi_{1,\nu} + \phi_{2,\mu}\phi_{2,\nu}) - g_{\mu\nu}\mathcal{L}. \tag{6}$$

Clearly the EMT consists of identity tensor $g_{\mu\nu}$ up to constant and two symmetric diad (or affinors) [6, 7] with gradient base vectors, i.e. $\phi_{i,\mu\nu} \equiv \phi_{i,\nu\mu}$. In index-free notation

$$T = 2\mathcal{L}'(\nabla\phi_1 \otimes \nabla\phi_1 + \nabla\phi_2 \otimes \nabla\phi_2) - \mathcal{L}. \tag{7}$$

For a real scalar field instead of two diad we have only one.

2.2. Einstein tensor decomposition

All space-times of the concerned class have an interval, which can be written in isotropic coordinates as follows [8, §23.2]

$$ds^2 = e^{\tau(r)}dt^2 - e^{\rho(r)-\tau(r)}(dx^2 + dy^2 + dz^2), \tag{8}$$
$$r^2 = x^2 + y^2 + z^2, \qquad x = x^1,\ y = x^2,\ z = x^3,\ c = 1,$$

so the Einstein tensor (calculated by RGTC package v. 3.2.5) is

$$G_{\mu\nu} = \frac{1}{4}\left[\begin{array}{c|c} e^{2\tau(r)-\rho(r)}C(r) & 0 \\ \hline 0 & B(r)\delta_{ij} + x^i x^j A(r) \end{array}\right], \tag{9}$$

$$A(r) = -\frac{2r(\rho'/r)' - \rho'^2 + 2\tau'^2}{r^2}, \qquad B(r) = \frac{2(r\rho')' + r\tau'^2}{r}, \tag{10}$$

$$C(r) = \frac{8}{r}(\tau' - \rho') - (\tau' - \rho')^2 + 4(\tau'' - \rho''), \tag{11}$$

where each prime denotes differentiation with respect to r.

To hold equations (1) we must find decomposition of the Einstein tensor in the form close to known decomposition of EMT (7):

$$G = \sum_{i=1}^{\mathbf{rank}\, G} A_{(i)} \otimes A_{(i)} + \lambda I \tag{12}$$

or in components

$$G^\mu_\nu = \sum_{i=1}^{\mathbf{rank}\, G} A^\mu_{(i)} A_{(i)\nu} + \lambda g^\mu_\nu. \tag{13}$$

For certain λ the sum may consist of less than 4 summands since diads can disappear or merge into multiple-identical tensor λI. For the former case matrix rank of G must be less than 4: $\mathbf{rank}\, G < 4$, or, equivalently, there is one or more zero eigenvalues of G. For the latter λ must be eigenvalue of matrix $||G^\mu_\nu||$ and a multiple eigenvalue can lower the upper limit of the sum by its order [9, theorems 2.9.4 and 2.4.3].

If the upper limit of the sum can be lowered to 2 than it is possible to generate the concerned space-time by a charged scalar field. If this limit can be reduced to 1 than the space-time can be attributed to some real scalar field. So the problem is reduced to the determination of eigenvalues and eigenvectors of square (generally non-symmetric) 4×4 matrix:

$$\det ||G^\mu_\nu - \lambda g^\mu_\nu|| = 0. \tag{14}$$

Algebraic calculations by Mathematica 5.2 shows that the Einstein tensor has in general one eigenvalue of order two

$$\lambda_I = \lambda_{II} = -\frac{B(r)}{4} e^{\tau(r)-\rho(r)}, \tag{15}$$

$$A^\mu_I = \{0, -z, 0, x\}, \qquad A^\mu_{II} = \{0, -y, x, 0\}, \tag{16}$$

and two eigenvalues of order one

$$\lambda_{III} = -\frac{r^2 A(r) + B(r)}{4} e^{\tau(r)-\rho(r)}, \qquad A^\mu_{III} = \{0, x, y, z\}, \tag{17}$$

$$\lambda_{IV} = \frac{C(r)}{4} e^{\tau(r)-\rho(r)}, \qquad A^\mu_{IV} = \{1, 0, 0, 0\}, \tag{18}$$

eigenvectors A_I and A_{II} being Killing vectors of spherical symmetry and λ_{III} corresponding to spherically-symmetric eigenvector field A_{III}. Hence, as expected for such type of the space-time symmetry [1, § 5.1], the Einstein tensor belongs to subtype a) of type I [(11)11] by Petrov classification [10, § 54].

2.3. Space-time determination

Since we consider the time-independent case, then vectors $\nabla\phi_{1,2}$ don't have the time-components $(\nabla\phi_{1,2})^0$. So there is only one possibility: the time-time-component of G^μ_ν (i. e. $G^0_0 = \lambda_{IV}$) must coincide with $-\mathcal{L}$. If we further recall that there are no rotation-invariant non-singular curl-free vector fields on the sphere (except the trivial zero one, see, for example, [8, Box 23.3]), then we have to require that

$$\lambda_I = \lambda_{II} = \lambda_{IV}. \tag{19}$$

In the opposite case either we cannot exclude a diad from G or after excision of radial diad from Einstein tensor we arrive to the sum of two pure spherical-tangent base vectors of diads.

The case considered then reduces to $B(r) + C(r) = 0$, so

$$\tau = \frac{\rho}{2} + \ln\frac{r}{r_0} - \int \left(\frac{e^{-\rho/2}}{r^2} \int e^{\rho/2}\, dr \right) dr. \tag{20}$$

The formula above gives us the most general form of spherically-symmetric static space-times with arbitrary Lagrangian scalar fields in minimally coupled case. It should be noted that the symmetry of space-time annihilates the difference between real and charged scalar field, so that any SSS space-time, originated by a complex scalar field, can be generated by a real scalar field only.

3. Ordinary Lagrangian with scalar field self-action

Now we investigate the case of generally accepted Lagrangian $\mathcal{L} = \frac{1}{2} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + V(\phi)$. Einstein equations have the form of $(\lambda_I = \lambda_{II} = \lambda_{IV} = \lambda)$

$$\frac{8\pi G}{c^4}(\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\mathcal{L}) = \lambda g_{\mu\nu} + (\lambda_{III} - \lambda)\frac{A_{III\,\mu}A_{III\,\nu}}{A_{III\,\alpha}A_{III}{}^{\alpha}}, \tag{21}$$

so we can find consequently:

$$\phi_{,\mu} = A_{III\,\mu}\sqrt{\frac{(\lambda_{III} - \lambda)c^4}{8\pi G(A_{III\,\alpha}A_{III}{}^{\alpha})}}, \tag{22}$$

and $\phi(r)$ by integration. Then the function $\phi(r)$ must be inverted to obtain $r(\phi)$ and finally by known $\mathcal{L}(r)$ we can find $V(\phi)$. Obviously there are many obstacles on this way, but it will be discussed elsewhere.

4. Conclusion

The family of SSS space-times, generated by a scalar field, depends on an arbitrary function $\rho(r)$, which uniquely determines both the space-time and Lagrangian. However Lagrangian is obtained not in the explicit form, but in the form of functional equations. Solution of the equations can be found in exceptional cases only, for example, in the case of ordinary Lagrangians considered above. From the other side, this solution family can be interpreted also as ideal fluid solutions, due to the same algebraic structure of EMT for an ideal fluid and a real scalar field. Moreover, cosmological constant in Einstein equations is not a problem for this method because it will be absorbed naturally into $-g_{\mu\nu}\mathcal{L}$ term.

In this paper we demonstrate a general method for account "the inverse problems" in General Relativity. This technique can be applied not only in such simple toy models but even for more relevant astrophysical problems of dark matter distribution, as it will be shown in our subsequent paper.

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