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Non-trivial analytical solutions in 2- and 3-dimensional scalar field models

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Here we demonstrate in explicit form the analytical solutions (instantons) for the scalar field models in $S^1 \otimes R^1$ space-time and analytical solutions (solitons) for the scalar field models in $S^1 \otimes R^1 \otimes R^1$ space-time.

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Let us consider 2-dimensional scalar model in $S^1 \otimes R^1$ space-time ($-\frac{L}{2} \le x \le \frac{L}{2}, -\infty < t < \infty$):

$$\mathrm{L} = \frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi - V(\varphi), \qquad \mu = 0, 1, \tag{1}$$

$$V(\varphi) = \lambda(1 - \cos(\rho\varphi)), \tag{1.1}$$

$$V(\varphi) = \lambda(\rho^2 - \varphi^2)^2, \tag{1.2}$$

where $\varphi(x,t)$, λ, ρ are real.

This models admit instantons (non-trivial classical solutions of the field equations in imaginary time $\tau = it$) [1]:

$$\varphi^{inst}(x,\tau) = \pm\frac{4}{\rho}\arctan e^{(\tau-\tau_0)\rho\sqrt{\lambda}}, \tag{2.1}$$

$$\varphi^{inst}(x,\tau) = \pm\tanh\left(\rho\sqrt{2\lambda}(\tau-\tau_0)\right), \tag{2.2}$$

where τ_0 is arbitrary parameter. These space-homogeneous solutions can be used for the description of the tunneling transitions between neighbor classically degenerated vacua of $V(\varphi)$ [2].

The (Euclidean) actions on instantons can be easy calculated [1]:

$$S_I[\varphi^{inst}(x,\tau)] = \frac{8\sqrt{\lambda}}{\rho}L, \tag{3.1}$$

$$S_I[\varphi^{inst}(x,\tau)] = \frac{4\sqrt{2\lambda}}{3}\rho^3 L. \tag{3.2}$$

Let us consider 3-dimensional scalar model in $S^1 \otimes R^1 \otimes R^1$ space-time ($-\frac{L}{2} \le x \le \frac{L}{2}, -\infty < y < \infty, -\infty < t < \infty$):

$$\mathrm{L} = \frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi - V(\varphi), \qquad \mu = 0, 1, 2, \tag{4}$$

where $V(\varphi)$ are defined in (1.1) and (1.2).

It is well-known, that instanton solutions in D dimensions formally coincide with static solitons in $D+1$ dimensions. Therefore we can immediately write exact solitons solutions:

$$\varphi^{sol}(x,y,t) = \pm\frac{4}{\rho}\arctan e^{(y-y_0)\rho\sqrt{\lambda}}, \tag{5.1}$$

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$$\varphi^{sol}(x,y,t) = \pm \tanh\left(\rho\sqrt{2\lambda}(y-y_0)\right). \tag{5.2}$$

The energies of solitons (5.1) and (5.2) formally coincide with (Euclidean) actions on instantons (3.1) and (3.2) correspondingly:

$$E[\varphi^{sol}(x,y,t)] = \frac{8\sqrt{\lambda}}{\rho}L, \tag{6.1}$$

$$E[\varphi^{sol}(x,y,t)] = \frac{4\sqrt{2\lambda}}{3}\rho^3 L. \tag{6.2}$$

Depending on time solitons can be obtained from solutions (5.1) and (5.2) by transition to another frame of reference:

$$\varphi^{sol}(x,y,t) = \pm\frac{4}{\rho}\arctan\left[\exp\left(\frac{(y-y_0)-ut}{\sqrt{1-u^2}}\rho\sqrt{\lambda}\right)\right], \tag{7.1}$$

$$\varphi^{sol}(x,y,t) = \pm\tanh\left[\frac{(y-y_0)-ut}{\sqrt{1-u^2}}\rho\sqrt{2\lambda}\right], \tag{7.2}$$

where u is a velocity of new frame of reference.

References

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[2] R.G.Shulyakovsky. *In Proceed. of the 13th Annual Seminar "Nonlinear Phenomena in Complex Systems"* (May 16-19, 2006, Minsk, Belarus), 240.