Nonlinear Dynamics and Applications. Vol. 14 (2007) 14 - 19

The equations for colour scalar particles in the field of the plane-wave collective excitations of a QGP

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Matrix equations for multiplet of colour scalar particles in the fields of collective excitations of a quark-gluon plasma are proposed. These equations are analyzed for a longitudinal non-abelian plane-wave propagating through the Quark-Gluon Plasma.

PACS numbers: 12.38.Mh, 52.25.Dg, 03.65.Pm,
Keywords: quark-gluon plasma, non-abelian plane wave, relativistic wave equations, covariant method

1. Introduction

The longwavelength excitations of a quark-gluon plasma at high temperature can be described as collective oscillations of gauge and fermionic average fields [1, 2].

At leading order in the gauge coupling g, ($g \ll 1$ in high temperature deconfined plasma), the collective dynamics is described entirely by a set of effective equations for soft gauge mean fields $A_\mu^a(x)$ (a are color indices for the adjoint representation of the gauge group) which describe long wavelength ($\eta \sim 1/gT$) and low frequency ($\omega \sim gT$) excitations (T denotes temperature). The equations satisfied by $A_\mu^a(x)$ are [2, 3]

$$\partial_\mu F_{\mu\nu}^a + igf^{abc}A_\mu^b F_{\mu\nu}^c = j_\mu^a, \tag{1}$$

where $F_{\mu\nu}^a = \partial_\mu A_\mu^a - \partial_\nu A_\mu^a - igf^{abc}A_\mu^b A_\nu^c$, f^{abc} are the structure constants of the $SU(N)$ group.

The induced current $j_\mu^a(x)$ describes the response of plasma to the color gauge fields $A_\mu^a(x)$. It is proportional to fluctuations in the phase-space color densities of quarks and gluons. Its expression for $SU(N)$ gauge group reads

$$j_\mu^a(x) = 3\omega_P^2 \int \frac{d\Omega}{4\pi}\, v_\mu W^a(x;v) \tag{2}$$

Here $\omega_P^2 = (2N + N_f)g^2T^2/18$ is the plasma frequency, N_f is the number of quark flavors, $v_\mu = (1, \mathbf{v})$, where $\mathbf{v} = \mathbf{q}/|\mathbf{q}|$ is the velocity of the hard particle with momentum $\mathbf{q}$, and the integral $\int d\Omega$ runs over all directions of the unit vector $\mathbf{v}$. Furthermore, the functions $W_a(x;v)$ are generally nonlocal and nonlinear functionals of the gauge fields $A_\mu(x)$ and are related to color polarizability of the plasma [3].

We note that, in contrast to the vacuum case, the solutions of the equations (1) for high temperature quark-gluon plasma have direct physical sense: they correspond to the collective color excitations of the QGP.

In this work we investigate the interaction of color scalar particles with the collective excitations of QGP. We use the five-component Duffin-Kemmer relativistic wave equations (RWE)

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to describe $SU(N)$ multiplets of scalar particles which interact with collective plane wave QGP excitation. We analyze this equations for $SU(2)$ longitudinal plane wave.

2. Covariant method for solving of relativistic wave equation in an external color gauge field

The equations for a multiplet of particles for an arbitrary spin in an external color vector field has the form [4, 5]

$$(\beta_\mu \partial_\mu + \beta_0)\Psi^k - ig\tau^a_{kn} A^a_\mu \Psi^n = 0, \tag{3}$$

where $\Psi^k = \Psi^k(x)$ is a multicomponent function, $x = \{x_\mu\} = \{x_0,\ \mathbf{x}\}$ is a 4-vector of space-time, β^μ and β^0 are constant square matrices, $\tau^a = \{\tau^a_{kn}\}$ are generators of the infinitesimal transformations of $SU(N)$ group in the space of functions Ψ^k, $([\tau^a\tau^b,\ \tau^b\tau^a] = if^{abc}\tau^c)$.

Let us describe the $SU(2)$ multiplet of scalar particles of mass m by means of the Duffin-Kemmer relativistic wave equations. In this case the multicomponent function $\Psi^k(x)$ has the form

$$\Psi^k = (\Psi^k_L(x)) = \begin{pmatrix} \Psi^k_0 \\ \Psi^k_\mu \end{pmatrix}, \tag{4}$$

where the function Ψ^k is transformed by means of the set of irreducible representations of the Lorentz group: $T = T^{00} \oplus T^{\frac{1}{2}\frac{1}{2}}$, and any irreducible representation of the $SU(2)$ group of the weight l. $([\tau^a\tau^b,\ \tau^b\tau^a] = i\varepsilon_{abc}\tau^c,\ \ \varepsilon_{abc}$ is the completely antisymmetric tensor, $\varepsilon_{123} = 1)$.

The matrices β^μ and β^0 have the form [6]

$$\beta^\mu = e^{0\mu} + e^{\mu 0}, \quad \beta^0 = m \otimes 1, \tag{5}$$

where e^{AB} (A, B$= 0, \mu$) are elements of the complete matrix algebra in the space of functions $\Psi^k(x)$.

Color vector fields A^a_μ are external fields of collective plane-wave excitations of QGP. Such solutions of the equations (1) and (2) have been found in paper [3] and have the form

$$A^a_\mu(x) = A^a_\mu(\varphi), \quad \partial_\mu A^a_\mu(x) = 0. \tag{6}$$

Here A^a_μ depends on x only through the variable $\varphi = k_\mu x_\mu = kx$, where $k = (ik_0, \mathbf{k})$ is a fixed, time-like, four-vector ($k^2 = -k_0^2 + \mathbf{k}^2 = -(\omega^2 - \mathbf{k}^2) = -\eta^2 < 0$). For the classical Yang-Mills equations in vacuum, plane-wave solutions ($k^2 = 0$) have been investigated in [7, 8]

Let us use the covariant method of solving of the RWE in external fields [9] for investigation of the equations (3). We find the solutions of the equations (3) for the multiplets of the scalar particles in the field of QGP excitations (5) in the form

$$\Psi^l(x) = \chi^l(\varphi)e^{ipx}, \tag{7}$$

where $p = (\mathbf{p}, ip_0)$ is the energy-momentum four-vector of scalar fields Ψ^l, $p^2 = -m^2$, $\varphi = kx = -k_0x_0 + \mathbf{kx}$, $kA^a = 0$

After substitution of (6) and (7) in (3) we obtain the equation for functions χ^l

$$\widehat{k}\chi'^l + (i\widehat{p} + m)\chi^l - ig(\tau^a)_{ln}\widehat{A}^a\chi^n = 0, \tag{8}$$

where $\widehat{A}^a = \beta^\mu A^a_\mu$.

The four-dimension functions $\chi^l(\varphi)$ and the result of acting the operator $\widehat{k}^a = k_\mu\beta^\mu$ on they are represented in the form

$$\chi^l = \begin{pmatrix} U^l_0 \\ U^l \end{pmatrix}, \qquad \widehat{k}\chi^l = \begin{pmatrix} kU^l \\ kU^l_0 \end{pmatrix}. \tag{9}$$

Here U_0^l is a multiplet of scalar functions, U^l is a multiplet of four-vector functions, $kU^l = k_\mu U_\mu^l$. Inserting (9) in (8) we obtain the following system of equations for the functions U_0^l, U^l:

$$\begin{aligned}&(kU'^l) + i(pU^l) + mU_0^l \\ &- ig(\tau^a)_{ln}\,(A^a U^n) = 0,\end{aligned} \tag{10}$$

$$U_0'^l k + mU^l + ipU_0^l - ig(\tau^a)_{ln} A^a U_0^n = 0. \tag{11}$$

where $U'^l = \partial_\varphi U'^l(\varphi),\ U_0'^l = \partial_\varphi U_0'^l(\varphi)$.

From the equations (10) and (11) we obtain the equations for the multiplet of scalar functions U_0^l. Multiplying the equations (10) by vector k, we found

$$kU^l = -\frac{i\mu}{m}U_0^l - \frac{k^2}{m}U_0'^l, \tag{12}$$

where $\mu = kp = -p_0k_0 + \mathbf{kp}$.

In turn, after multiplication of (12) on vector k we obtain

$$U^l = -\frac{i\mu}{m}\frac{k}{k^2}U_0^l - \frac{1}{m}kU'^{\,l}_{\,0}. \tag{13}$$

The multiplication of expression (11) on vector p leads to the equation

$$ipU^l = -\frac{i\mu}{m}U'^{\,l}_{\,0} - mU_0^l - \frac{g}{m}(\tau^a)_{ln}(pA^a)U_0^n. \tag{14}$$

By means of analogous transformations we obtain

$$U_0'^l k + mU^l + ipU_0^l - ig(\tau^a)_{ln} A^a U_0^n = 0, \tag{15}$$

$$\begin{aligned}&A^a U^l = \\ &-\frac{1}{m}\left\{i(pA^a)\delta_{lm} - ig(\tau^d)_{lm}(A^a A^d)\right\} U_0^m.\end{aligned} \tag{16}$$

Using the equations (13)-(16) we obtain from the equation (12) the second-order differential equations for the multiplet of the scalar functions U_0^l

$$\begin{aligned}&\mathcal{D}U_0^l = G_{ln}U_0^l, \\ &\mathcal{D} = (\partial_{\varphi\varphi} + \kappa\partial_\varphi), \quad \kappa = -\frac{2i\mu}{\eta^2}, \\ &G_{ln} = \frac{2g}{\eta^2}(\tau^a)_{ln}(pA^a) \\ &\qquad - \frac{g^2}{\eta^2}(\tau^a)_{ln}(\tau^d)_{nm}(A^a A^d)\end{aligned} \tag{17}$$

After substitution

$$U_0^l(\varphi) = e^{(-\frac{\kappa}{2}\varphi)}\Phi_0^l(\varphi)), \tag{18}$$

we obtain the equations for function Φ_l^0 in the form

$$\begin{aligned}\Phi''^{\,l}_{\,0} =&\Gamma_{ln}\Phi_0^l, \\ \Gamma_{ln} =&G_{ln} + \frac{\mu^2}{\eta^4}\delta_{ln} \\ &= \frac{\mu^2}{\eta^4}\delta_{ln} + \frac{2g}{\eta^2}(\tau^a)_{ln}(pA^a) \\ &- \frac{g^2}{\eta^2}(\tau^a)_{ln}(\tau^d)_{nm}(A^a A^d).\end{aligned} \tag{19}$$

The equations (19) are the second-order differential equations, in contrast to the first-order ones which were solved in [4, 5] for multiplet of scalar particles in the external massless ($k^2 = 0$) non-Abelian plane-wave fields.

3. Colour scalar particles in longitudinal $SU(2)$ plane-wave excitations of QGP

Let us investigate the equations (19) for longitudinal $SU(2)$ plane-wave excitations of QGP. Such solutions of the equation (1) and (2) has been obtained in the paper [3] in the form

$$\begin{gathered} A^1 = 0, \quad A^2 = 0, \quad A^3_\mu(\varphi) = e^3_\mu h_3(\varphi) \\ \left\{e^3_\mu(k)\right\} = \frac{1}{\sqrt{\omega^2 - \mathbf{k}^2}} \left\{|\mathbf{k}|, \omega \frac{\mathbf{k}}{|\mathbf{k}|}\right\} \end{gathered} \tag{20}$$

where the function $h_3(\varphi)$ satisfies the harmonic oscillator equations

$$(\omega^2 - \mathbf{k}^2) h_3'' + \Omega_L^2 h_3 = 0, \tag{21}$$

and has the form

$$\begin{aligned} & h_3(\varphi) = C_1 \cos(\nu_L \varphi) + C_1 \sin(\nu_L \varphi), \\ & \nu_L = \Omega_L / \sqrt{\omega^2 - \mathbf{k}^2}, \\ & \Omega_L^2 = 3\omega_P \frac{\omega^2 - \mathbf{k}^2}{\mathbf{k}^2} \left(Q\left(\frac{\omega}{|\mathbf{k}|}\right) - 1 \right). \end{aligned} \tag{22}$$

Here C_1 and C_2 are integration constants, $Q(u) = (u/2)\ln((u+1)/(u-1))$.

After substitution of (20) in (19) we obtain

$$\begin{aligned} \Phi''^{\,l}_{\;0} =& \Gamma_{ln} \Phi^l_0, \\ \Gamma_{ln} =& \frac{\mu^2}{\eta^4} \delta_{ln} + \frac{2g}{\eta^2} (\tau^3)_{ln} (pe^3) h_3(\varphi) \\ & - \frac{g^2}{\eta^2} (\tau^3)^2_{ln} h_3^2(\varphi) \end{aligned} \tag{23}$$

To eliminate τ^3 we can use the projective operator method [9]. The characteristic equation of matrices τ^3 is written as

$$\begin{aligned} p^l(\tau^3) = & [(\tau^3)^2 - l^2][(\tau^3)^2 - (l-1)^2] \ldots \\ & \ldots \left[(\tau^3)^2 - \left(\frac{3 + (-1)^{2l}}{4} \right) \right] \\ & \times (\tau^3)^{\frac{1}{2}(1 + (-1)^{2l})} = 0. \end{aligned} \tag{24}$$

where l is the $SU(2)$ representation weight.

The projective operator for the eigenvalues $\lambda = -l, -l+1, \dots, l-1, l$ of the matrices τ^3 is $\varrho_\lambda = Q^\lambda(\tau^3)/Q^\lambda(\lambda)$. Here $Q^\lambda(\tau^3)$ is the truncated minimal polynomial,

$$\begin{aligned} Q^\lambda(\tau^3) = {} & [(\tau^3)^2 - l^2][(\tau^3)^2 - (l-1)^2]\dots \\ & \dots[(\tau^3)^2 - (\lambda+1)^2][(\tau^3)^2 - (\lambda-1)^2]\dots \\ & \dots\left[(\tau^3)^2 - \left(\frac{3+(-1)^{2l}}{4}\right)\right] \\ & \quad \times (\tau^3)^{\frac{1}{2}(1+(-1)^{2l})}. \end{aligned} \tag{25}$$

After multiplication of ϱ_λ on (23) and using the relations $\varrho_\lambda \tau^3 = \lambda \varrho_\lambda$ and $\varrho_\lambda (\tau^3)^2 = \lambda^2 \varrho_\lambda$ we obtain the second order equation without matrices τ^3:

$$\Phi''^{\,l}_{\lambda}(\varphi) = \Gamma(\lambda, \varphi)\Phi^l_\lambda(\varphi), \tag{26}$$

$$\Gamma(\lambda, \varphi) = \frac{\mu^2}{\eta^4} + \frac{2g}{\eta^2}\lambda(pe^3)h_3(\varphi) \tag{27}$$

$$- \frac{g^2}{\eta^2}\lambda^2 h_3^2(\varphi) \tag{28}$$

where $\Phi^l_\lambda(\varphi) = \varrho_\lambda \Phi^l_0(\varphi)$

The function Ψ^k is expressed through the function $\Phi^l_\lambda(\varphi)$ as follows:

$$U^l_0(\varphi) = e^{(-\frac{\kappa}{2}\varphi)} \sum_\lambda \Phi^l_\lambda(\varphi) \tag{29}$$

$$U^l(\varphi) = -\frac{i\mu}{m}\frac{k}{\eta^2}U^l_0(\varphi) - \frac{1}{m}kU'^{\,l}_{0}(\varphi). \tag{30}$$

Because of the functions $\Gamma(\lambda, \varphi)$ in the equations (26) are periodic, the functions $\Phi^l_\lambda(\varphi)$ satisfy the ordinary differential Hill equations. Solutions of such equations can be found in the form of convergent series [10].

4. Conclusions

In this paper we have investigated the equations for a SU(2) multiplet of colour scalar particles in the fields of the plane-wave excitations of QGP. The covariant method of solving RWE equations are used to analyze these equations. For $SU(2)$ color group the equations for multiplet of scalar particles in the field of the longitudinal plane-wave excitations of QGP are divided into a set of independent ordinary second-order differential Hill-equations, which depend on single variable $\varphi = k_\mu x_\mu$.

Acknowledgments

This research was supported in part by the BFFR-Grant F06MC-020 (Minsk, Belarus).

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