Nonlinear Dynamics and Applications. Vol. 14 (2007) 126 - 139

Maxwell equations in Riemannian space-time, geometry effect on material equations in media

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In the paper, the known possibility to consider the (vacuum) Maxwell equations in a curved space-time as Maxwell equations in flat space-time (Mandel'stam L.I., Tamm I.E. [1,2]) as taken in an effective media the properties of which are determined by metrical structure of the initial curved model $g_{\alpha\beta}(x)$ is studied

$$H^{\rho\sigma}(x) = \sqrt{-g(x)}\; g^{\rho\alpha}(x) g^{\sigma\beta}(x) \; \Big[\; \epsilon_0 \; \Delta_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu}(x) \; \Big] \; ;$$

${\Delta_{\alpha\beta}}^{ab}$ – 4-rank tensor; metrical structure of the curved space-time generates effective constitutive equations for electromagnetic fields:

$$\mathbf{D} = \epsilon_0 \; \epsilon(x) \; \mathbf{E} + \epsilon_0 c \; \alpha(x) \; \mathbf{B} \; , \qquad \mathbf{H} = \epsilon_0 c \; \beta(x) \; \mathbf{E} + \frac{1}{\mu_0} \; \mu(x) \; \mathbf{B} \; ,$$

the form of four symmetrical tensors $\epsilon^{ik}(x), \alpha^{ik}(x), \beta^{ik}(x), \mu^{ik}(x)$ is found explicitly for general case of an arbitrary Riemannian space-time geometry $g_{\alpha\beta}(x)$:

$$\epsilon^{ik}(x) = \sqrt{-g}\; [g^{00}(x) g^{ik}(x) - g^{0i}(x) g^{0k}(x)], \qquad \alpha^{ik}(x) = +\sqrt{-g}\; g^{ij}(x)\; g^{0l}(x)\; \epsilon_{ljk},$$

$$\beta^{ik}(x) = -\; \sqrt{-g}\; g^{0j}(x)\; \epsilon_{jil}\; g^{lk}(x) \; , \qquad \mu^{ik}(x) = \sqrt{-g}\; \frac{1}{2} \epsilon_{imn} g^{ml}(x) g^{nj}(x) \epsilon_{ljk} \; .$$

The main peculiarity of the geometrical generating for effective electromagnetic medias characteristics consists in the following: four tensors $\epsilon^{ik}(x), \alpha^{ik}(x), \beta^{ik}(x), \mu^{ik}(x)$ are not independent and obey some additional constraints between them.Several, the most simple examples are specified in detail: it is given geometrical modeling of the anisotropic media (magnetic crystals) and the geometrical modeling of a uniform media in moving reference frame in the background of Minkowski electrodynamics – the latter is realized trough the use of a non-diagonal metrical tensor determined by 4-vector velocity of the moving uniform media $g^{am} = [\; g^{am} + (\epsilon\mu - 1)\; u^a u^m \;] / \sqrt{\mu}$. Also the effective material equations generated by geometry of space of constant curvature (Lobachevsky and Riemann models) are determined. General problem of geometrical transforming arbitrary (linear) material equations, given by $\epsilon^{(0)}, \alpha^{(0)}, \beta^{(0)}, \mu^{(0)}$, has been studied – corresponding formulas have been produced.

PACS numbers: 03.50.De, 04.20.-q
Keywords: Maxwell equations, media, constitutive relations, Riemannian geometry

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1. Riemannian geometry and Maxwell theory

Let us start with the Maxwell equations in Minkowski space: in vector notation they are [3-6]

$$(I) \qquad \text{div } \mathbf{B} = 0 \;, \qquad \text{rot } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \;,$$

$$(II) \qquad \epsilon\epsilon_0 \text{ div } \mathbf{E} = \rho \;, \qquad \frac{1}{\mu\mu_0} \text{ rot } \mathbf{B} = \mathbf{J} + \epsilon\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \;. \tag{1}$$

With the use of material equations

$$\mathbf{H} = \frac{\mathbf{B}}{\mu\mu_0} \;, \qquad \mathbf{D} = \epsilon\epsilon_0 \;\mathbf{E} \tag{2}$$

eqs. (1) can be written in terms of four vectors as follows

$$(I) \qquad \text{div } c\mathbf{B} = 0 \;, \qquad \text{rot } \mathbf{E} = -\frac{\partial c\mathbf{B}}{\partial x^0} \;,$$

$$(II) \qquad \text{div } \mathbf{D} = j^0 \;, \qquad \text{rot } \frac{\mathbf{H}}{c} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial x^0} \tag{3}$$

where $x^0 = ct$, $j^a = (\rho, \mathbf{J}/c)$, In terms of two electromagnetic tensors:

$$(F^{\alpha\beta}) = \left| \begin{array}{cccc} 0 & -E^1 & -E^2 & -E^3 \\ +E^1 & 0 & -cB^3 & +cB^2 \\ +E^2 & +cB^3 & 0 & -cB^1 \\ +E^3 & -cB^2 & +cB^1 & 0 \end{array} \right| \;, \; (H^{\alpha\beta}) = \left| \begin{array}{cccc} 0 & -D^1 & -D^2 & -D^3 \\ +D^1 & 0 & -H^3/c & +H^2/c \\ +D^2 & +H^3/c & 0 & -H^1/c \\ +D^3 & -H^2/c & +H^1/c & 0 \end{array} \right|$$

eqs. (3) take the form

$$(I) \qquad \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \;, \qquad (II) \qquad \partial_b H^{ba} = j^a \;. \tag{4}$$

In vacuum case, the material equations (note the notation $E^i = -E_i,\; D^i = -D_i,\; B^i = +B_i,\; H^i = +H_i$)

$$\mathbf{D} = \epsilon_0 \mathbf{E} = (D^i) \;, \qquad \mathbf{H} = \frac{1}{\mu_0} \;\mathbf{B} = (H^i),$$

will look in tensor form as follows:

$$H^{ab}(x) = \epsilon_0 \; F^{ab}(x) \;.$$

The situation is quite different in non-vacuum case. For instance, the material equations for a uniform media

$$\mathbf{D} = \epsilon_0 \epsilon \mathbf{E} = (D^i) \;, \qquad \mathbf{H} = \frac{1}{\mu_0 \mu} \;\mathbf{B} = (H^i),$$

these relationships can be written in short form with the help of subsidiary 4×4 - matrix

$$\eta^{am} = \sqrt{\epsilon} \left| \begin{array}{cccc} 1/k & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \\ 0 & 0 & -k & 0 \\ 0 & 0 & 0 & -k \end{array} \right| \;, \qquad k = \frac{1}{\sqrt{\epsilon\mu}}, \qquad H^{ab} = \epsilon_0 \; \eta^{am} \eta^{bn} \; F_{mn} \tag{5}$$

When extending Maxwell theory to the case of space-time with non-Euclidean geometry, which can describe gravity according to General Relativity [6], one must change previous equations to a more general form [6] (for simplicity, let us start with the most simple case of vacuum Maxwell equations):

$$(I) \qquad \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0 \;,$$

$$(II) \qquad \nabla_\beta H^{\beta\alpha} = j^\alpha \;, \qquad H_{\alpha\beta} = \epsilon_0 \; F_{\alpha\beta} \;. \tag{6}$$

2. Maxwell equations in Riemannian space-time and a media

Let us discuss in detail the known possibility [1-2] to consider the (vacuum) Maxwell equations in a curved space-time as Maxwell equations in flat space-time but taken in an effective media the properties of which are determined by metrical structure of the initial curved model $g_{\alpha\beta}(x)$. Let us restrict ourselves to the case of curved space-time models which are parameterized by the same quasi-Cartesian coordinate system x^a.

Vacuum Maxwell equations in a Riemannian space-time, parameterized by the same quasi-Cartesian coordinates (to distinguish formulas referring to a flat and curved models let us use small letters to designates electromagnetic tensors in curved model, f_{ab} and h^{ab})

$$(I) \qquad \partial_a f_{bc} + \partial_b f_{ca} + \partial_c f_{ab} = 0 \; , \qquad (II) \qquad \frac{1}{\sqrt{-g}} \, \partial_b \, \sqrt{-g} \; f^{ba} = \frac{1}{\epsilon_0} j^a \; . \tag{7}$$

One can immediately see that introducing new (formal) variables (there exists one special case; namely, if $g(x)$ does not depend on coordinates in fact then the factor $\sqrt{-g}$ can be omitted from the formulas and below)

$$F_{ab} = f_{ab}, \qquad H^{ba} = \epsilon_0 \, \sqrt{-g} \; g^{am}(x) g^{bn}(x) \; f_{mn}(x), \qquad \sqrt{-g} \; j^a \longrightarrow j^a \tag{8}$$

equations (7) in the curved space can be re-written as Maxwell equations of the type (**??**) in flat space but in a media:

$$(I) \qquad \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \qquad (II) \qquad \partial_b \; F^{ba} = \frac{1}{\epsilon_0} j^a \; . \tag{9}$$

At this, relations playing the role of material equations are determined by metrical structure:

$$H^{\beta\alpha}(x) = \epsilon_0 \; [\; \sqrt{-g(x)} \; g^{\alpha\rho}(x) g^{\beta\sigma}(x) \;] \; F_{\rho\sigma}(x) \; ; \tag{10}$$

if $g_{\alpha\beta}$ does not depend upon coordinates, then the factor $\sqrt{-g(x)}$ can be omitted — see (8).

3. Metrical tensor $g_{\alpha\beta}(x)$ and material equations

In this section let us consider the material equations for electromagnetic fields which are generated by metrical structure of the curved space-time model. Consider the case of arbitrary metrical tensor

$$g_{\alpha\beta}(x) = \left| \begin{array}{cccc} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{array} \right| . \tag{11}$$

We are to obtain a 3-dimensional form of relation (10). Their general structure should be as follows (for discussion of different types of electromagnetic medias see in [7-12]):

$$\begin{aligned} D^i &= \epsilon_0 \; \epsilon^{ik}(x) \; E_k + \epsilon_0 c \; \alpha^{ik}(x) \; B_k \; , \\ H^i &= \epsilon_0 c \; \beta^{ik}(x) \; E_k + \frac{1}{\mu_0} \; \mu^{ik}(x) \; B_k \; . \end{aligned} \tag{12}$$

Four dimensionless (3×3)-matrices $\epsilon^{ik}(x)$, $\alpha^{ik}(x)$, $\beta^{ik}(x)$, $\mu^{ik}(x)$ should not be independent because they are bilinear functions of 10 independent components of the symmetrical tensor

$g_{\alpha\beta}(x)$. After simple calculation, one produces expressions for four tensors:

$$\begin{aligned} \epsilon^{ik}(x) &= \sqrt{-g}\ (g^{00}(x)\ g^{ik}(x) - g^{0i}(x)\ g^{0k}(x))\ , \\ \mu^{ik}(x) &= \frac{1}{2}\ \sqrt{-g}\ \epsilon_{imn}\ g^{ml}(x) g^{nj}(x)\ \epsilon_{ljk}\ , \\ \alpha^{ik}(x) &= +\sqrt{-g}\ g^{ij}(x)\ g^{0l}(x)\ \epsilon_{ljk}\ , \\ \beta^{ik}(x) &= -\sqrt{-g}\ g^{0j}(x)\ \epsilon_{jil}\ g^{lk}(x)\ . \end{aligned} \tag{13}$$

The above form the tensors obey special symmetry conditions:

$$\epsilon^{ik}(x) = +\epsilon^{ki}(x)\ , \qquad \mu^{ik}(x) = +\mu^{ki}(x)\ , \qquad \beta^{ki}(x) = \alpha^{ik}\ ; \tag{14}$$

which mean that the (6×6)-matrix defining material equations

$$\left| \begin{array}{c} D^i(x) \\ H^i(x) \end{array} \right| = \left| \begin{array}{cc} \epsilon_0\ \epsilon^{ik}(x) & \epsilon_0 c\ \alpha^{ik}(x) \\ \epsilon_0 c\ \beta^{ik}(x) & \mu_0^{-1}\ \mu^{ik}(x) \end{array} \right| \left| \begin{array}{c} E_k(x) \\ B_k(x) \end{array} \right| \tag{15}$$

is a symmetrical matrix. Four (material) tensor in the above formulas are defined by

$$[\ \epsilon^{ik}(x)\ ] = \sqrt{-g}\ g^{00} \left| \begin{array}{ccc} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{array} \right| - \sqrt{-g} \left| \begin{array}{ccc} g^1\ g^1 & g^1\ g^2 & g^1\ g^3 \\ g^2\ g^1 & g^2\ g^2 & g^2\ g^3 \\ g^3\ g^1 & g^3\ g^2 & g^3\ g^3 \end{array} \right| ,$$

$$\mu^{ik}(x) = (\sqrt{-g} \left| \begin{array}{ccc} (g^{22}g^{33} - g^{23}g^{32}) & (g^{31}g^{23} - g^{21}g^{33}) & (g^{21}g^{32} - g^{22}g^{31}) \\ (g^{32}g^{13} - g^{33}g^{12}) & (g^{33}g^{11} - g^{31}g^{13}) & (g^{31}g^{12} - g^{32}g^{11}) \\ (g^{12}g^{23} - g^{13}g^{22}) & (g^{13}g^{21} - g^{11}g^{23}) & (g^{11}g^{22} - g^{12}g^{21}) \end{array} \right| ,$$

$$\alpha^{ik}(x) = \sqrt{-g} \left| \begin{array}{ccc} (-g^{12}g^3 + g^{13}g^2) & (g^{11}g^3 - g^{13}g^1) & (-g^{11}g^2 + g^{12}g^1) \\ (-g^{22}g^3 + g^{23}g^2) & (g^{21}g^3 - g^{23}g^1) & (-g^{21}g^2 + g^{22}g^1) \\ (-g^{32}g^3 + g^{33}g^2) & (g^{31}g^3 - g^{33}g^1) & (-g^{31}g^2 + g^{32}g^1) \end{array} \right| ,$$

$$\beta^{ik}(x) = \sqrt{-g} \left| \begin{array}{ccc} (-g^{12}g^3 + g^{13}g^2) & (-g^{22}g^3 + g^{23}g^2) & (-g^{32}g^3 + g^{33}g^2) \\ (g^{11}g^3 - g^{13}g^1) & (g^{21}g^3 - g^{23}g^1) & (g^{31}g^3 - g^{33}g^1) \\ (-g^{11}g^2 + g^{12}g^1) & (-g^{21}g^2 + g^{22}g^1) & (-g^{31}g^2 + g^{32}g^1) \end{array} \right| . \tag{16}$$

4. Geometrical modeling of the uniform media

Let us consider one special form of the metrical tensor:

$$g_{\alpha\beta}(x) = \left| \begin{array}{cccc} a^2 & 0 & 0 & 0 \\ 0 & -b^2 & 0 & 0 \\ 0 & 0 & -b^2 & 0 \\ 0 & 0 & 0 & -b^2 \end{array} \right| , \tag{17}$$

where a^2 and b^2 are arbitrary (positive) numerical parameters. This is a special case mentioned in connection with eq. (8): if $g(x)$ does not depend on coordinates in fact then the factor $\sqrt{-g}$ can be omitted from the formulas. Acting so we get the material equations generated by that geometry

$$(\epsilon^{ik}) = \frac{1}{a^2 b^2} \left| \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array} \right| , \qquad (\mu^{ik}) = \frac{1}{b^4} \left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right| , \tag{18}$$

or differently

$$D^i = -\frac{\epsilon_0}{a^2 b^2}\, E_i \;, \qquad H^i = \frac{1}{\mu_0 b^4}\, B_i \;, \tag{19}$$

from which it follows

$$b^2 = \sqrt{\mu} \;, \qquad a^2 = \frac{1}{\epsilon}\frac{1}{\sqrt{\mu}} \;. \tag{20}$$

Corresponding metrical tensor (17) is

$$g_{\alpha\beta}(x) = \frac{1}{\sqrt{\epsilon}} \left| \begin{array}{cccc} 1/\sqrt{\epsilon\mu} & 0 & 0 & 0 \\ 0 & -\sqrt{\epsilon\mu} & 0 & 0 \\ 0 & 0 & -\sqrt{\epsilon\mu} & 0 \\ 0 & 0 & 0 & -\sqrt{\epsilon\mu} \end{array} \right| . \tag{21}$$

5. Geometrical modeling of an anisotropic media

Let us extend the previous analysis and consider another metrical tensor:

$$g_{\alpha\beta} = \left| \begin{array}{cccc} a^2 & 0 & 0 & 0 \\ 0 & -b_1^2 & 0 & 0 \\ 0 & 0 & -b_2^2 & 0 \\ 0 & 0 & 0 & -b_3^2 \end{array} \right| , \tag{22}$$

where a^2, b_1^2, b_2^2, b_3^2, are arbitrary numerical parameters. The material equations generated by that geometry are

$$D^i = \epsilon_0 \epsilon^{ik}\, E_k \;, \qquad (\epsilon^{ik}) = a^{-2} \left| \begin{array}{ccc} -b_1^{-2} & 0 & 0 \\ 0 & -b_2^{-2} & 0 \\ 0 & 0 & -b_3^{-2} \end{array} \right| , \tag{23}$$

$$H^i = \mu_0^{-1} \mu^{ik}\, B_k \;, \qquad (\mu^{ik}) = \left| \begin{array}{ccc} b_2^{-2} b_3^{-2} & 0 & 0 \\ 0 & b_3^{-2} b_1^{-2} & 0 \\ 0 & 0 & b_1^{-2} b_2^{-2} \end{array} \right| ,$$

or differently

$$D^1 = -\frac{\epsilon_0}{a^2 b_1^2}\, E_1 \;, \qquad D^2 = -\frac{\epsilon_0}{a^2 b_2^2}\, E_2 \;, \qquad D^3 = -\frac{\epsilon_0}{a^2 b_3^2}\, E_3 \;,$$

$$H^1 = \frac{1}{\mu_0\; b_2^2 b_3^2}\, B_1 \;, \qquad H^2 = \frac{1}{\mu_0\; b_3^2 b_1^2}\, B_2 \;, \qquad H^3 = \frac{1}{\mu_0\; b_1^2 b_2^2}\, B_3 \;.$$

These material equations should be compared with

$$D^1 = -\epsilon_0 \epsilon_1\, E_1 \;, \qquad D^2 = -\epsilon_0 \epsilon_2\, E_2 \;, \qquad D^3 = -\epsilon_0 \epsilon_3\, E_3 \;,$$

$$H^1 = \frac{1}{\mu_0 \mu_1}\, B_1 \;, \qquad H^2 = \frac{1}{\mu_0 \mu_2}\, B_2 \;, \qquad H^3 = \frac{1}{\mu_0 \mu_3}\, B_3 \;,$$

from which it follows

$$\epsilon_1 = \frac{1}{a^2 b_1^2} \;, \qquad \epsilon_2 = \frac{1}{a^2 b_2^2} \;, \qquad \epsilon_3 = \frac{1}{a^2 b_3^2} \;,$$

$$\mu_1 = b_2^2\, b_3^2 \;, \qquad \mu_2 = b_3^2\, b_1^2 \;, \qquad \mu_3 = b_1^2\, b_2^2 \;. \tag{24}$$

One can readily obtain

$$\frac{\mu_1}{\epsilon_1} = \frac{\mu_2}{\epsilon_2} = \frac{\mu_3}{\epsilon_3} = (a^2\ b_1^2\ b_2^2\ b_3^2) = -g\ ,$$

$$-g = \sqrt{\frac{\mu_1^2 + \mu_2^2 + \mu_3^2}{\epsilon_1^2 + \epsilon_2^2 + \epsilon_2^2}}\ , \qquad \frac{\mu_i}{\sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2}} = \frac{\epsilon_i}{\sqrt{\epsilon_1^2 + \epsilon_2^2 + \epsilon_2^2}}\ . \tag{25}$$

The latter means that one may use four independent parameters, ϵ, μ, n_i:

$$\epsilon_i = \epsilon\ n_i\ , \qquad \mu_i = \mu\ n_i, \qquad \mathbf{n}^2 = 1 \tag{26}$$

One can readily express b_i^2 in terms of μ_i:

$$\begin{aligned}
\mu_2\mu_3 = b_1^4\ (b_2^2\ b_3^2) = b_1^4\ \mu_1 \qquad &\Longrightarrow \qquad b_1^2 = \sqrt{\frac{\mu_2\mu_3}{\mu_1}} = \sqrt{\mu}\ \sqrt{\frac{n_2 n_3}{n_1}}\ ,\\
\mu_3\mu_1 = b_2^4\ (b_3^2\ b_1^2) = b_2^4\ \mu_2 \qquad &\Longrightarrow \qquad b_2^2 = \sqrt{\frac{\mu_3\mu_1}{\mu_2}} = \sqrt{\mu}\ \sqrt{\frac{n_3 n_1}{n_2}}\ ,\\
\mu_1\mu_2 = \mu_0^2\ b_3^4\ (b_1^2\ b_2^2) = \mu_0\ b_3^4\ \mu_3 \qquad &\Longrightarrow \qquad b_3^2 = \sqrt{\frac{\mu_1\mu_2}{\mu_3}} = \sqrt{\mu}\ \sqrt{\frac{n_1 n_2}{n_3}}\ .
\end{aligned} \tag{27}$$

In turn, from $a^2\ b_1^2\ b_2^2\ b_3^2 = \mu/\epsilon$ it follows

$$a^2 = \frac{\mu}{\epsilon}\ \frac{1}{b_1^2 b_2^2 b_3^2} = \frac{1}{\epsilon\sqrt{\mu}}\ \frac{1}{\sqrt{n_1 n_2 n_3}} \tag{28}$$

The formula (27)-(28) provide us with (anisotropic) extension

$$g_{ab}(x) = \frac{1}{\sqrt{\epsilon}} \left| \begin{array}{cccc}
\frac{1}{\sqrt{\epsilon\mu}}\ \frac{1}{\sqrt{n_1 n_2 n_3}} & 0 & 0 & 0 \\
0 & -\sqrt{\epsilon\mu}\ \sqrt{\frac{n_2 n_3}{n_1}} & 0 & 0 \\
0 & 0 & -\sqrt{\epsilon\mu}\ \sqrt{\frac{n_3 n_1}{n_2}} & 0 \\
0 & 0 & 0 & -\sqrt{\epsilon\mu}\ \sqrt{\frac{n_1 n_2}{n_3}}
\end{array} \right| \tag{29}$$

of the previous (isotropic) metrical tensor.

6. The moving media and anisotropy

One other, more involved, example of effective anisotropic media is provided by the material equations for uniform media for a moving observer (more details see in [14-16]):

$$\Delta^{abmn} = \frac{\epsilon_0}{\mu}\ [\ g^{am} + (\epsilon\mu - 1)\ u^a u^m\ ]\ [\ g^{bn} + (\epsilon\mu - 1)\ u^b u^n\ ]\ , \qquad H^{ab}(x) = \Delta^{abmn}\ F_{mn}\ . \tag{30}$$

Corresponding four 3-dimensional tensors are

$$\epsilon^{ik} = \frac{1}{\mu} \left| \begin{array}{ccc} (-1+\gamma u^1 u^1 - \gamma u^0 u^0) & \gamma u^1 u^2 & \gamma u^1 u^3 \\ \gamma u^1 u^2 & (-1+\gamma u^2 u^2 - \gamma u^0 u^0) & \gamma u^2 u^3 \\ \gamma u^3 u^1 & \gamma u^3 u^2 & (-1+\gamma u^3 u^3 - \gamma u^0 u^0) \end{array} \right| ,$$

$$\mu^{ik} = \frac{1}{\mu} \left| \begin{array}{ccc} (1-\gamma u^2 u^2 - \gamma u^3 u^3) & \gamma u^1 u^2 & \gamma u^1 u^3 \\ \gamma u^1 u^2 & (1-\gamma u^3 u^3 - \gamma u^1 u^1) & \gamma u^2 u^3 \\ \gamma u^3 u^1 & \gamma u^3 u^2 & (1-\gamma u^1 u^1 - \gamma u^2 u^2) \end{array} \right| ,$$

$$\alpha^{ik} = \frac{1}{\mu} \left| \begin{array}{ccc} 0 & -\gamma u^0 u^3 & +\gamma u^0 u^2 \\ +\gamma u^0 u^3 & 0 & -\gamma u^0 u^1 \\ -\gamma u^0 u^2 & +\gamma u^0 u^1 & 0 \end{array} \right| , \qquad \beta^{ik} = \frac{1}{\mu} \left| \begin{array}{ccc} 0 & +\gamma u^0 u^3 & -\gamma u^0 u^2 \\ -\gamma u^0 u^3 & 0 & +\gamma u^0 u^1 \\ +\gamma u^0 u^2 & -\gamma u^0 u^1 & 0 \end{array} \right| . \qquad (31)$$

Let us deduce 3-dimensional vector form of these relations. For the vector D^i we have

$$D^1 = \frac{\epsilon_0}{\mu} \left[(-1+\gamma u^1 u^1 - \gamma u^0 u^0) E_1 + \gamma u^1 u^2 E_2 + \gamma u^1 u^3 E_3 \right] + \frac{\epsilon_0 c}{\mu} (-\gamma u^0 u^3 B_2 + \gamma u^0 u^2 B_3)$$
$$D^2 = \frac{\epsilon_0}{\mu} \left[+\gamma u^1 u^2 E_1 + (-1+\gamma u^2 u^2 - \gamma u^0 u^0) E_2 + \gamma u^2 u^3 E_3 \right] + \frac{\epsilon_0 c}{\mu} (\gamma u^0 u^3 B_1 - \gamma u^0 u^1 B_3)$$
$$D^1 = \frac{\epsilon_0}{\mu} \left[+\gamma u^1 u^3 E_1 + \gamma u^2 u^3 E_2 + (-1+\gamma u^3 u^3 - \gamma u^0 u^0) E_3 \right] + \frac{\epsilon_0 c}{\mu} (-\gamma u^0 u^2 B_1 + \gamma u^0 u^1 B_2)$$

and further

$$D^1 = -\frac{\epsilon_0}{\mu} E_1 + \frac{\epsilon_0 \gamma}{\mu} \left[-u^0 u^0 E_1 + (u^1 E_1 + u^2 E_2 + u^3 E_3)\, u^1 \right] + \frac{\epsilon_0 c \gamma}{\mu} u^0 \, (u^2 B_3 - u^3 B_2) ,$$
$$D^2 = -\frac{\epsilon_0}{\mu} E_2 + \frac{\epsilon_0 \gamma}{\mu} \left[-u^0 u^0 E_2 + (u^1 E_1 + u^2 E_2 + u^3 E_3)\, u^2 \right] + \frac{\epsilon_0 c \gamma}{\mu} u^0 \, (u^3 B_1 - u^1 B_3) ,$$
$$D^3 = -\frac{\epsilon_0}{\mu} E_3 + \frac{\epsilon_0 \gamma}{\mu} \left[-u^0 u^0 E_3 + (u^1 E_1 + u^2 E_2 + u^3 E_3)\, u^3 \right] + \frac{\epsilon_0 c \gamma}{\mu} u^0 \, (u^1 B_2 - u^2 B_1) ,$$

With the use of notation

$$u^0 = \frac{1}{\sqrt{1-V^2}}, \qquad u^i = \frac{V^i}{\sqrt{1-V^2}}$$

previous relations look as follows

$$D^1 = -\frac{\epsilon_0}{\mu} E_1 + \frac{\epsilon_0 \gamma}{\mu} \frac{\left[-E_1 + (V^1 E_1 + V^2 E_2 + V^3 E_3)\, V^1 \right]}{1-V^2} + \frac{\epsilon_0 c \gamma}{\mu} \frac{(V^2 B_3 - V^3 B_2)}{1-V^2} ,$$
$$D^2 = -\frac{\epsilon_0}{\mu} E_2 + \frac{\epsilon_0 \gamma}{\mu} \frac{\left[-E_1 + (V^1 E_1 + V^2 E_2 + V^3 E_3)\, V^2 \right]}{1-V^2} + \frac{\epsilon_0 c \gamma}{\mu} \frac{(V^3 B_1 - V^1 B_3)}{1-V^2} ,$$
$$D^3 = -\frac{\epsilon_0}{\mu} E_3 + \frac{\epsilon_0 \gamma}{\mu} \frac{\left[-E_3 + (V^1 E_1 + V^2 E_2 + V^3 E_3)\, V^3 \right]}{1-V^2} + \frac{\epsilon_0 c \gamma}{\mu} \frac{(V^1 B_2 - V^2 B_1)}{1-V^2} ,$$

or in vector form they are

$$\mathbf{D} = +\frac{\epsilon_0}{\mu} \mathbf{E} + \frac{\epsilon_0 \gamma}{\mu} \frac{\mathbf{E} - (\mathbf{VE})\,\mathbf{V}}{1-V^2} + \frac{\epsilon_0 c \gamma}{\mu} \frac{\mathbf{V} \times \mathbf{B}}{1-V^2} , \qquad (32)$$

Now analogously we should consider three relations for H^i:

$$H_1 = \frac{1}{\mu_0 \mu} \left[(1-\gamma u^2 u^2 - \gamma u^3 u^3)\, B_1 + \gamma u^1 u^2 B_2 + \gamma u^1 u^3 B_3 \right] + \frac{\epsilon_0 c}{\mu} (\gamma u^0 u^3 E_2 - \gamma u^0 u^2 E_3)$$
$$H_2 = \frac{1}{\mu_0 \mu} \left[\gamma u^1 u^2 B_1 + (1-\gamma u^3 u^3 - \gamma u^1 u^1)\, B_2 + \gamma u^2 u^3 B_3 \right] + \frac{\epsilon_0 c}{\mu} (\gamma u^0 u^3 E_2 - \gamma u^0 u^2 E_3)$$
$$H_3 = \frac{1}{\mu_0 \mu} \left[\gamma u^3 u^1 B_1 + \gamma u^3 u^2 B_3 + (1-\gamma u^1 u^1 - \gamma u^2 u^2)\, B_3 \right] + \frac{\epsilon_0 c}{\mu} (\gamma u^0 u^2 E_1 - \gamma u^0 u^1 E_2)$$

or further

$$H_1 = \frac{1}{\mu_0 \mu} \, B_1 + \frac{\gamma}{\mu_0 \mu} (-u^2 u^2 \, B_1 - u^3 u^3 \, B_1 + u^1 u^2 B_2 + u^1 u^3 B_3 \,) + \frac{\epsilon_0 c \, \gamma}{\mu} \, u^0 \, (u^3 E_2 - u^2 E_3)$$
$$H_2 = \frac{1}{\mu_0 \mu} \, B_2 + \frac{\gamma}{\mu_0 \mu} (+u^1 u^2 B_1 - u^3 u^3 \, B_2 - u^1 u^1 \, B_2 + u^2 u^3 B_3 \,) + \frac{\epsilon_0 c \, \gamma}{\mu} \, u^0 \, (u^3 E_2 - u^2 E_3)$$
$$H_3 = \frac{1}{\mu_0 \mu} \, B_3 + \frac{\gamma}{\mu_0 \mu} (+u^3 u^1 B_1 + u^3 u^2 B_2 - u^1 u^1 \, B_3 - u^2 u^2 \, B_3 \,) + \frac{\epsilon_0 c \, \gamma}{\mu} \, u^0 \, (u^2 E_1 - u^1 E_2)$$

Noting identities

$$(-u^2 u^2 \, B_1 - u^3 u^3 \, B_1 + u^1 u^2 B_2 + u^1 u^3 B_3 \,) =$$
$$= u^2 (u^1 B_2 - u^2 \, B_1) - u^3 (u^3 B_1 - u^1 B_3) = [\mathbf{u} \times (\mathbf{u} \times \mathbf{B})]_1$$
$$(+u^1 u^2 B_1 - u^3 u^3 \, B_2 - u^1 u^1 \, B_2 + u^2 u^3 B_3 \,) =$$
$$= u^3 (u^2 B_3 - u^3 \, B_2) - u^1 (u^1 B_2 - u^2 \, B_1) = [\mathbf{u} \times (\mathbf{u} \times \mathbf{B})]_2$$
$$(+u^3 u^1 B_1 + u^3 u^2 B_2 - u^1 u^1 \, B_3 - u^2 u^2 \, B_3 \,) =$$
$$= u^1 (u^3 B_1 - B_1 U^3)) - u^2 (u^2 \, B_3 - u^3 B_2) = [\mathbf{u} \times (\mathbf{u} \times \mathbf{B})]_3$$

previous relations can be rewritten in a vector form as follows:

$$\mathbf{H} = \frac{1}{\mu_0 \mu} \, \mathbf{B} + \frac{\gamma}{\mu_0 \mu} \frac{\mathbf{V} \times (\mathbf{V} \times \mathbf{B}))}{1 - V^2} + \frac{\epsilon_0 c \, \gamma}{\mu} \, \frac{\mathbf{V} \times \mathbf{E}}{1 - V^2} \tag{33}$$

Relations (32)-(33) provide us with 3-dimensional vector form of material equations in media moving with velocity $\mathbf{V}$. Firstly they were produced by H. Minkowski.

7. Effective material equations generated by Riemannian geometry of a space of constant positive curvature

A 3-dimensional space of constant positive curvature, S_3, has many applications in physical problems. The most simple realization of this model is given by three-sphere in 4-space (the space of the unitary group $SU(2)$):

$$W_4^2 + W_1^2 + W_2^2 + W_3^2 = R^2 \, , \qquad w_1 = \frac{W_1}{R}, \quad \text{and so on} \; ; \tag{34}$$

R is a curvature radius. These four coordinate are connected with quasi-spherical ones be relations:

$$w_4 = \cos \chi, \qquad w_i = \sin \chi \; n_i \; , \qquad n_i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \sin \theta) \; . \tag{35}$$

The most used coordinates are conformally-flat ones:

$$y^i = \frac{2 w_i}{1 + w_4} = 2 \tan \chi / 2 \; n_i \; ,$$
$$dl^2 = R^2 \; (1 + \frac{y^2}{4})^{-2} \; [dy_1^2 + dy_2^2 + dy_3^2 \,] \tag{36}$$

and quasi-Cartesian ones :

$$x^i = \frac{y^i}{1 - y^2/4} = \tan \chi \; n_i = \frac{w_i}{w_4} \; , \qquad \chi \in [0, \pi/2] \; ; \tag{37}$$

the later parameterize only elliptical model (the space of orthogonal group $S0(3)$)

$$dS^2 = c^2 dt^2 - (\frac{\delta_{jk}}{1+x^2} - \frac{x^j x^k}{(1+x^2)^2})\, dx^j dx^j \; ,$$

$$g_{\alpha\beta} = \begin{vmatrix} 1 & 0 \\ 0 & g_{jk} \end{vmatrix}, \qquad g_{jk} = -(\frac{\delta_{jk}}{1+x^2} - \frac{x^j x^k}{(1+x^2)^2}) \; ,$$

$$g^{\alpha\beta} = \begin{vmatrix} 1 & 0 \\ 0 & g^{kl} \end{vmatrix}, \qquad g^{kl} = -(1+x^2)(\delta_{kl} + x^k x^l) \; . \qquad (38)$$

Also calculating the determinant

$$\text{det}\,(g_{\alpha\beta}) = \frac{1}{\text{det}\,(g^{\alpha\beta})}, \qquad \text{det}\,(g^{\alpha\beta}) = -(1+x^2)^3 \begin{vmatrix} 1+x^1x^1 & x^1x^2 & x^1x^3 \\ x^2x^1 & 1+x^2x^2 & x^2x^3 \\ x^3x^1 & x^3x^2 & 1+x^3x^3 \end{vmatrix} =$$

$$-(1+x^2)^3\,[(1+x^1x^1)(1+x^2x^2)(1+x^3x^3) + 2x^1x^1x^2x^2x^3x^3 -$$

$$-(1+x^2x^2)x^1x^1x^3x^3 - (1+x^1x^1)x^2x^2x^3x^3 - (1+x^3x^3)x^1x^1x^2x^2] = -(1+x^2)^3 \; , \qquad (39)$$

that is

$$\sqrt{-\text{det}\,(g_{\alpha\beta})} = \frac{1}{(1+x^2)^{3/2}} \; . \qquad (40)$$

For effective dielectric tensor $\epsilon^{ik}(x)$ we have

$$\epsilon^{ik}(x) = \sqrt{-g}\; g^{00}(x) g^{ik}(x) = -\frac{1}{\sqrt{1+x^2}}(\delta_{ik} + x^i x^k) =$$

$$= -\frac{1}{\sqrt{1+x^2}} \begin{vmatrix} 1+x^1x^1 & x^1x^2 & x^1x^3 \\ x^1x^2 & 1+x^2x^2 & x^2x^3 \\ x^3x^1 & x^3x^2 & 1+x^3x^3 \end{vmatrix} \; . \qquad (41)$$

For effective magnetic tensor $\mu^{ik}(x)$ we have

$$\mu^{ik}(x) = \sqrt{1+x^2} \begin{vmatrix} (1+x^2x^2+x^3x^3) & -x^1x^2 & -x^1x^3 \\ -x^2x^1 & (1+x^3x^3+x^1x^1) & -x^2x^3 \\ -x^3x^1 & -x^3x^2 & (1+x^1x^1+x^2x^2) \end{vmatrix} \qquad (42)$$

It is easily verified by direct calculation that (taken with minus) dielectric tensor $(-\epsilon^{ik}(x))$ and tensor $\mu^{ik}(x)$ are inverse to each other

$$-\epsilon^{ik}(x)\; \mu^{kl}(x) = \delta_{ik} \; . \qquad (43)$$

Let us write down the effective material equations explicitly

$$D^i = \epsilon_0 \epsilon^{ik} E_k \; , \qquad H^i = \frac{1}{\mu_0}\mu^{ik} B_k, \qquad B_i = \mu_0 M^{ik} H^k \; ; \qquad (44)$$

at this two matrices coincide

$$-\epsilon^{ik}(x) = M^{ik}(x) = \frac{1}{\sqrt{1+x^2}} \begin{vmatrix} 1+x^1x^1 & x^1x^2 & x^1x^3 \\ x^1x^2 & 1+x^2x^2 & x^2x^3 \\ x^3x^1 & x^3x^2 & 1+x^3x^3 \; , \end{vmatrix} \qquad (45)$$

8. Effective material equations generated by Lobachevsky geometry of a space of constant negative curvature

A 3-dimensional space of constant negative curvature, H_3, has many applications in physical problems. The most simple realization of this model is given by three-sphere in 4-space (the space of the unitary group $SU(1.1)$):

$$W_4^2 - W_1^2 - W_2^2 - W_3^2 = R^2 \, , \qquad w_1 = \frac{W_1}{R}, \quad \text{and so on} \; ; \tag{46}$$

R is a curvature radius. These four coordinate are connected with quasi-spherical ones be relations:

$$w_4 = \cosh\chi, \qquad w_i = \sinh\chi \, n_i \; , \qquad n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \sin\theta) \; , \chi \in [0, +\infty) \; . \tag{47}$$

The most used coordinates are conformally-flat ones:

$$y^i = \frac{2w_i}{1+w_4} = 2\tanh\chi/2 \; n_i \; ,$$
$$dl^2 = R^2 \; (1 - \frac{y^2}{4})^{-2} \; [dy_1^2 + dy_2^2 + dy_3^2 \;] \tag{48}$$

quasi-Cartesian ones :

$$x^i = \frac{y^i}{1 + y^2/4} = \tanh\chi \; n_i = \frac{w_i}{w_4} \; , \qquad \chi \in [0, \pi/2] \; ; \tag{49}$$

the later parameterize only elliptical model (the space of orthogonal group $S0(3)$)

$$dS^2 = c^2 dt^2 - (\frac{\delta_{jk}}{1-x^2} + \frac{x^j x^k}{(1-x^2)^2}) \; dx^j dx^j \; ,$$
$$g_{\alpha\beta} = \left| \begin{array}{cc} 1 & 0 \\ 0 & g_{jk} \end{array} \right| , \qquad g_{jk} = -(\frac{\delta_{jk}}{1-x^2} + \frac{x^j x^k}{(1-x^2)^2}) \; ,$$
$$g^{\alpha\beta} = \left| \begin{array}{cc} 1 & 0 \\ 0 & g^{kl} \end{array} \right| , \qquad g^{kl} = -(1-x^2)(\delta_{kl} - x^k x^l) \; . \tag{50}$$

Also calculating the determinant

$$\det \, (g_{\alpha\beta}) = \frac{1}{\det \, (g^{\alpha\beta})}, \qquad \det \, (g^{\alpha\beta}) = -(1-x^2)^3 \; , \tag{51}$$

that is

$$\sqrt{-\det \, (g_{\alpha\beta})} = \frac{1}{(1-x^2)^{3/2}} \; . \tag{52}$$

For effective dielectric tensor $\epsilon^{ik}(x)$ we have

$$\epsilon^{ik}(x) = \sqrt{-g} \; g^{00}(x) g^{ik}(x) = -\frac{1}{\sqrt{1-x^2}} (\delta_{ik} - x^i x^k =$$
$$= -\frac{1}{\sqrt{1-x^2}} \left| \begin{array}{ccc} 1 - x^1x^1 & -x^1x^2 & -x^1x^3 \\ -x^1x^2 & 1 - x^2x^2 & -x^2x^3 \\ -x^3x^1 & -x^3x^2 & 1 - x^3x^3 \end{array} \right| \; . \tag{53}$$

For effective magnetic tensor $\mu^{ik}(x)$ we have

$$\mu^{ik}(x) = \sqrt{-g} \left| \begin{array}{ccc} (g^{22}g^{33} - g^{23}g^{32}) & (g^{31}g^{23} - g^{21}g^{33}) & (g^{21}g^{32} - g^{22}g^{31}) \\ (g^{32}g^{13} - g^{33}g^{12}) & (g^{33}g^{11} - g^{31}g^{13}) & (g^{31}g^{12} - g^{32}g^{11}) \\ (g^{12}g^{23} - g^{13}g^{22}) & (g^{13}g^{21} - g^{11}g^{23}) & (g^{11}g^{22} - g^{12}g^{21}) \end{array} \right| =$$

$$= \sqrt{1-x^2} \left| \begin{array}{ccc} (1 - x^2x^2 - x^3x^3) & x^1x^2 & x^1x^3 \\ x^2x^1 & (1 - x^3x^3 - x^1x^1) & x^2x^3 \\ x^3x^1 & x^3x^2 & (1 - x^1x^1 - x^2x^2) \end{array} \right| \qquad (54)$$

It is easily verified by direct calculation that (taken with minus) dielectric tensor $(-\epsilon^{ik}(x))$ and tensor $\mu^{ik}(x)$ are inverse to each other

$$-\epsilon^{ik}(x)\ \mu^{kl}(x) = \delta_{ik}\ . \qquad (55)$$

Let us write down the effective material equations explicitly

$$D^i = \epsilon_0 \epsilon^{ik} E_k\ , \qquad H^i = \frac{1}{\mu_0} \mu^{ik} B_k, \qquad B_i = \mu_0 M^{ik} H^k\ ; \qquad (56)$$

at this two matrices coincide

$$-\epsilon^{ik}(x) = M^{ik}(x) = \frac{1}{\sqrt{1-x^2}} \left| \begin{array}{ccc} 1 - x^1x^1 & -x^1x^2 & -x^1x^3 \\ -x^1x^2 & 1 - x^2x^2 & -x^2x^3 \\ -x^3x^1 & -x^3x^2 & 1 - x^3x^3\ , \end{array} \right| . \qquad (57)$$

9. Geometry effect on material equations in media

Above, we have started with Maxwell equations in vacuum:

$$\begin{array}{c} \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0\ , \\ \partial_b H^{ba} = j^a\ , \qquad H_{ab} = \epsilon_0\ F_{ab} \end{array} \qquad (58)$$

and changed them to generally covariant in Riemannian space-time

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0\ , \qquad \frac{1}{\sqrt{-g}}\ \partial_\beta \sqrt{-g}\ H^{\beta\alpha} = j^\alpha\ . \qquad (59)$$

At this, vacuum material equations

$$H_{\alpha\beta} = \epsilon_0\ F_{\alpha\beta}, \qquad (60)$$

due to presence of metrical tensor $g^{\rho\alpha}(x)$ gave us the modified (effective) material equations

$$H^{\rho\sigma}(x) = \sqrt{-g}\ g^{\rho\alpha}(x) g^{\sigma\beta}(x)\ \epsilon_0\ F_{\alpha\beta}(x)\ . \qquad (61)$$

As a first generalization let us start with Maxwell equations in a uniform media:

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0\ , \qquad \partial_b H^{ba} = j^a\ , \qquad (62)$$

$$H_{mn} = \epsilon_0\ \eta_m^{\ a}\ \eta_n^{\ b}\ \ F_{ab}, \qquad \eta_m^{\ a} = \sqrt{\epsilon} \left| \begin{array}{cccc} k^{-1} & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{array} \right|\ , \ \ k^2 = \frac{1}{\epsilon\mu}\ . \qquad (63)$$

Extension of these to Riemannian space-time looks as

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \; , \qquad \frac{1}{\sqrt{-g}} \, \partial_\beta \sqrt{-g} \; H^{\beta\alpha} = j^\alpha \; , \tag{64}$$

At this, material equations for the uniform media

$$H_{\alpha\beta}(x) = \epsilon_0 \; \eta_\alpha^{\;\;a} \; \eta_\beta^{\;\;b} \; F_{ab}(x) \; , \qquad \eta_\alpha^{\;\;a} = \sqrt{\epsilon} \left| \begin{array}{cccc} k^{-1} & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{array} \right| \; . \tag{65}$$

will take the form

$$\begin{array}{c} H^{\rho\sigma}(x)(x) = \sqrt{-g} \; g^{\rho\alpha}(x) g^{\sigma\beta}(x) \; H_{\alpha\beta}(x) = \\ = \sqrt{-g} \; g^{\rho\alpha}(x) g^{\sigma\beta}(x) \; \epsilon_0 \;\; \eta_\alpha^{\;\;a} \; \eta_\beta^{\;\;b} \;\; F_{ab}(x) \; . \end{array} \tag{66}$$

With the notation

$$\hat{F}_{\alpha\beta}(x) = \eta_\alpha^{\;\;a} \; \eta_\beta^{\;\;b} \;\; F_{ab}(x) \; ;$$

they are written as follows

$$H^{\rho\sigma}(x)(x) = \sqrt{-g} \; g^{\rho\alpha}(x) g^{\sigma\beta}(x) \; \epsilon_0 \; \hat{F}_{\alpha\beta}(x) \; . \; . \tag{67}$$

where explicit form of $\hat{F}_{\alpha\beta}(x)$ is

$$\hat{F}_{\alpha\beta}(x) = \left| \begin{array}{cc} 0 & \epsilon F_{0i} \\ \epsilon F_{i0} & \mu^{-1} F_{ik} \end{array} \right| . \tag{68}$$

One should not make any additional calculation, instead it suffices the make one formal change $F_{\alpha\beta}(x) \Longrightarrow \hat{F}_{\alpha\beta}(x)$, and now material equations are

$$\begin{array}{c} D^i = \epsilon_0 \epsilon \; \epsilon^{ik}(x) \; E_k + \epsilon_0 \epsilon c \; \alpha^{ik}(x) \; B_k \; , \\ H^i = \epsilon_0 \epsilon c \; \beta^{ik}(x) \; E_k + \dfrac{1}{\mu_0 \mu} \; \mu^{ik}(x) \; B_k \; . \end{array} \tag{69}$$

These relations provide us with material equations for uniform media modified by Riemannian geometry of background space-time.

It is easily to make one other extension: let us start with anisotropic media in Minkowski space

$$D_i = \epsilon_0 \; \epsilon_{kl}^{(0)} E_l \; , \qquad H_i = \frac{1}{\mu_0} \; \mu_{kl}^{(0)} B_k \; . \tag{70}$$

they will be modified into

$$\begin{array}{c} D^i = \epsilon_0 \; [\epsilon^{ik}(x) \; \epsilon_{kl}^{(0)}] \;\; E_l + \epsilon_0 c \; [\alpha^{ik}(x) \; \mu_{kl}^{(0)}] \;\; B_l \; , \\ H^i = \epsilon_0 c \; [\beta^{ik}(x) (\epsilon_{kl}^{(0)}] \;\; E_l + \dfrac{1}{\mu_0} \; [\mu^{ik}(x) (\mu_{kl}^{(0)}] \;\; B_l \; . \end{array} \tag{71}$$

And now, final extension: let start with arbitrary (linear) media when material equations are determined by 4-rank tensor

$$H_{\alpha\beta}(x) = \epsilon_0 \; \Delta_{\alpha\beta}^{\;\;\;\;ab} \; F_{ab}(x) \tag{72}$$

from which Riemannian geometry will generate the following ones

$$H^{\rho\sigma}(x)(x) = \sqrt{-g}\; g^{\rho\alpha}(x) g^{\sigma\beta}(x)\; \epsilon_0\; \Delta_{\alpha\beta}{}^{ab}\; F_{ab}(x)\; . \tag{73}$$

or in 3-dimensional form

$$\begin{aligned} D^i &= \epsilon_0\; \epsilon^{ik}(x)\; \Big[\epsilon^{(0)}_{kl}\; E_l + c\alpha^{(0)}_{kl}\; B_l\Big] + \epsilon_0\; \alpha^{ik}(x)\; \Big[\beta^{(0)}_{kl}\; E_l + \mu^{(0)}_{kl} c\; B_l\Big]\;, \\ H^i &= \epsilon_0 c\; \beta^{ik}(x)\; \Big[\epsilon^{(0)}_{kl}\; E_l + c\alpha^{(0)}_{kl}\; B_l\Big] + \epsilon_0 c\; \mu^{ik}(x)\; \Big[\beta^{(0)}_{kl}\; E_l + \mu^{(0)}_{kl} c\; B_l\Big]\;; \end{aligned} \tag{74}$$

these may be rewritten differently

$$\begin{aligned} D^i &= \epsilon_0\; \Big[\epsilon^{ik}(x)\; \epsilon^{(0)}_{kl} + \alpha^{ik}(x)\beta^{(0)}_{kl}\Big]\; E_l + \epsilon_0 c\; \Big[\epsilon^{ik}(x)\alpha^{(0)}_{kl} + \alpha^{ik}(x)\mu^{(0)}_{kl}\Big]\; B_l\;, \\ H^i &= \epsilon_0 c\; \Big[\beta^{ik}(x)\epsilon^{(0)}_{kl} + \mu^{ik}(x)\beta^{(0)}_{kl}\Big]\; E_l + \frac{1}{\mu_0}\Big[\beta^{ik}(x)\alpha^{(0)}_{kl} + \mu^{ik}(x)\mu^{(0)}_{kl}\Big]\; B_l\;, \end{aligned}$$

or in matrix form (with no indices)

$$\begin{aligned} \mathbf{D} &= \epsilon_0\; \big[\epsilon(x)\; \epsilon^{(0)} + \alpha(x)\beta^{(0)}\big]\; \mathbf{E} + \epsilon_0 c\; \big[\epsilon(x)\alpha^{(0)} + \alpha(x)\mu^{(0)}\big]\; \mathbf{B}\;, \\ \mathbf{H} &= \epsilon_0 c\; \big[\beta(x)\epsilon^{(0)} + \mu(x)\beta^{(0)}\big]\; \mathbf{E} + \frac{1}{\mu_0}\big[\beta(x)\alpha^{(0)} + \mu(x)\mu^{(0)}\big]\,\mathbf{B}\;. \end{aligned} \tag{75}$$

These formulas can be read symbolically:

$$\begin{aligned} \epsilon^0 &\Longrightarrow \hat{\epsilon} = \epsilon(x)\; \epsilon^{(0)} + \alpha(x)\beta^{(0)}, \qquad & \alpha^0 &\Longrightarrow \hat{\alpha} = \epsilon(x)\alpha^{(0)} + \alpha(x)\mu^{(0)}\;, \\ \beta^0 &\Longrightarrow \hat{\beta} = \beta(x)\epsilon^{(0)} + \mu(x)\beta^{(0)}\;, & \mu^0 &\Longrightarrow \hat{\mu} = \beta\alpha^{(0)} + \mu(x)\mu^{(0)}\;. \end{aligned} \tag{76}$$

For instance, if starting material equations have only diagonal blocks, that is $\alpha^0 = 0, \beta^0 = 0$, last relations become simpler:

$$\begin{aligned} \epsilon^0 &\Longrightarrow \hat{\epsilon} = \epsilon(x)\; \epsilon^{(0)}, \qquad & \alpha^0 = 0 &\Longrightarrow \hat{\alpha} = +\alpha(x)\mu^{(0)}\;, \\ \beta^0 = 0 &\Longrightarrow \hat{\beta} = \beta(x)\epsilon^{(0)}\;, & \mu^0 &\Longrightarrow \hat{\mu} = \mu(x)\mu^{(0)}\;. \end{aligned} \tag{77}$$

Four (material) tensor in the above formulas are defined by

This work was supported by Fund for Basic Research of Belarus F07-314. Authors are grateful to participants of seminar of Laboratory of Physics of Fundamental Interaction, National Academy of Sciences of Belarus, for discussion and advice.

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