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Quaternion formulation of nonlinear equations of the non-commutative electrodynamics

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The Lorenz covariant biquaternion formulation of the nonlinear equations of non-commutative electrodynamics is constructed. That opens an opportunity for dimensional reduction of obtained system.

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As is known, interest to models with non-commuting space-time coordinates has obtained a new impulse after occurrence of work by E. Witten and N. Seiberg [1] who have opened an opportunity of comparison to corresponding equations of their nonlinear analogues in usual space-time. Within the framework of such approach, electrodynamics without sources has been constructed in [2] whose equations possess, in opinion of authors, the property of the generalized dual invariance. As the considered equations look as system of the equations of classical macroscopical electrodynamics with the nonlinear material equations determined by antisymmetric non-commutative tensor of coordinates, it is pertinent to take advantage of a biquaternionic formalism to demonstrate symmetry properties of the equations [3]. Besides, the biquaternion formulation will allow using to the full advantages the Fedorov vector parameterization of the Lorenz group [4] and will open opportunities for direct use of covariant methods [5] to compute non-commutativity effects in optical processes.

As has been shown in work [2], the non-commutative generalization of usual Maxwell-Lagrange density

$$L = -\frac{1}{4}\hat{F}_{\mu\nu} * \hat{F}^{\mu\nu}$$

involves the star product of the non-commutative field strength $\hat{F}_{\mu\nu}$ obtained from the potential $\hat{A}_\mu$ as

$$\hat{F}^{\mu\nu} = \partial^\mu \hat{A}^\nu - \partial^\nu \hat{A}^\mu - ig\left(\hat{A}^\mu * \hat{A}^\nu - \hat{A}_\nu * \hat{A}_\mu\right),$$

where $g = \frac{e}{\hbar c}$and the star product is defined by $(f * g)(x) = e^{\frac{1}{2}\theta^{\alpha\beta}\partial_\alpha\partial'_\beta} f(x)\, g(x')\,|_{x=x'}$.

After the Witten–Seiberg transformation, it is possible to obtain equations of Maxwell macroscopical electrodynamics

$$\frac{1}{c}\frac{\partial}{\partial t}\underline{B} + [\nabla \underline{E}] = 0, (\nabla \underline{B}) = 0 \tag{1}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\underline{D} - [\nabla \underline{H}] = 0, (\nabla \underline{D}) = 0, \tag{2}$$

with the nonlinear material equations

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$$\underline{D}=\underline{E}-\left(\left(\theta \underline{B}\right)-\left(\varepsilon \underline{E}\right)\right)\underline{E}+\left(\left(\theta \underline{E}\right)+\left(\varepsilon \underline{B}\right)\right)\underline{B}+\left(\underline{E}\underline{B}\right)\underline{\theta}+\frac{1}{2}\left(\underline{E}^2-\underline{B}^2\right)\underline{\varepsilon} \quad (3)$$

$$\underline{H}=\underline{B}-\left(\left(\theta \underline{B}\right)-\left(\varepsilon \underline{E}\right)\right)\underline{B}+\left(\left(\theta \underline{E}\right)+\left(\varepsilon \underline{B}\right)\right)\underline{E}-\left(\underline{E}\underline{B}\right)\underline{\varepsilon}+\frac{1}{2}\left(\underline{E}^2-\underline{B}^2\right)\underline{\theta} \quad (4)$$

where $\underline{E}$ and $\underline{B}$ are electric and magnetic field, and $\underline{D}$ and $\underline{H}$ determine inductions of these fields respectively. Non-commutativity of space-time coordinates is determined by some anti-symmetric second rank tensor $\theta^{\mu\nu}$ satisfying the condition $[x^\mu x^\nu]=i\theta^{\mu\nu}$. Thus $\underline{\varepsilon}$ and $\underline{\theta}$ are determined from the conditions:

$$\varepsilon^i=\theta^{0i},\theta^i=\frac{1}{2}\xi^{ijk}\theta_{jk}.$$

Dual transformations for the equations (1-4) can be carried out by the replacements

$$\underline{E}\to-\underline{H},\underline{B}\to\underline{D},\underline{D}\to-\underline{B},\underline{H}\to\underline{E},\underline{\varepsilon}\to-\underline{\theta},\underline{\theta}\to-\underline{\varepsilon}. \quad (5)$$

Let us present now the equation (1-4) with the help of biquaternions. Recall for this purpose that any element of algebra of biquaternions in "vector" basis can be written as

$$q=q_0+\underline{q}=q_0e_0+q_a\underline{n}_a=q_s+\underline{q}_v,$$

where $\underline{n}_a e_0=e_0\underline{n}_a=\underline{n}_a$, $e_0^2=1$, $(\underline{n}_a\underline{n}_b)=\delta_{ab}n_an_b$, $[\underline{n}_a\underline{n}_b]=\varepsilon_{abc}\underline{n}_c$, $a,b,c=1,2,3$, e_0 and $\underline{n}_a$ are arbitrary generations elements of the quaternion algebra, and q_0 , q_a are elements of a field of complex numbers. Whence the law of multiplication of biquaternions in the vector form follows:

$$qp=q_0p_0-(\underline{q}\underline{p})+q_0\underline{p}+p_0\underline{q}+[\underline{q}\underline{p}].$$

For the biquaternions operations of complex conjugation $q^*=q_0^*-\underline{q}^*$ and quaternion conjugation $\bar{q}=q_0-\underline{q}$ are determined. The first operation is the automorphism, while the second one is the anti-automorphism of the biquaternion algebra $\overline{pq}=\bar{q}\cdot\bar{p}$.

In biquaternions the equations (1-4) is possible to present as

$$\nabla\left(\underline{B}-i\underline{E}\right)-\overline{\nabla\left(\underline{B}-i\underline{E}\right)^*}=0,\quad \nabla\left(\underline{H}-i\underline{D}\right)+\overline{\nabla\left(\underline{H}-i\underline{D}\right)^*}=0 \quad (6)$$

with the nonlinear material equations

$$\underline{B}-i\underline{E}=\underline{H}-i\underline{D}+\left[\left(\underline{H}+i\underline{D}\right)\left(\theta+i\varepsilon\right)\right]_s\left(\underline{H}-i\underline{D}\right)+\frac{1}{2}\left(\underline{H}+i\underline{D}\right)_s^2\left(\theta-i\varepsilon\right) \quad (7)$$

or

$$\underline{H}-i\underline{D}=\underline{B}-i\underline{E}-\left[\left(\underline{B}+i\underline{E}\right)\left(\theta+i\varepsilon\right)\right]_s\left(\underline{B}-i\underline{E}\right)-\frac{1}{2}\left(\underline{B}+i\underline{E}\right)_s^2\left(\theta-i\varepsilon\right). \quad (8)$$

To study the dual symmetry of the obtained equations it is necessary to present the equations (8-9) through pairs $(\underline{H}-i\underline{E})$, $(\underline{B}-i\underline{D})$. For the equations (8) it is possible to write down:

$$\nabla\left[\left(\underline{B}\mp i\underline{D}\right)\pm\left(\underline{H}\mp i\underline{E}\right)\right]-\left[\left(\underline{B}\mp i\underline{D}\right)\mp\left(\underline{H}\mp i\underline{E}\right)\right]\nabla=0. \quad (9)$$

For the equations of binding constraint (7) or (8) it is useful to sum them, since the equations are invariant with respect to the replacements (5):

$$\begin{gathered}\left[\left(\left(\underline{H}-i\underline{E}\right)-\left(\underline{B}-i\underline{D}\right)\right)\left(\underline{\theta}+i\underline{\varepsilon}\right)\right]\left(\underline{H}-i\underline{E}\right)+\left[\left(\left(\underline{H}+i\underline{E}\right)+\left(\underline{B}+i\underline{D}\right)\right)\left(\underline{\theta}+i\underline{\varepsilon}\right)\right]\left(\underline{H}+i\underline{E}\right)\\ +\left[\left(\left(\underline{H}-i\underline{E}\right)-\left(\underline{B}-i\underline{D}\right)\right)\left(\underline{\theta}+i\underline{\varepsilon}\right)\right]\left(\underline{B}-i\underline{D}\right)-\left[\left(\left(\underline{H}+i\underline{E}\right)+\left(\underline{B}+i\underline{D}\right)\right)\left(\underline{\theta}+i\underline{\varepsilon}\right)\right]\left(\underline{B}+i\underline{D}\right)\\ =\left[\left(\left(\underline{H}-i\underline{E}\right)-\left(\underline{B}-i\underline{D}\right)\right)\left(\left(\underline{H}+i\underline{E}\right)+\left(\underline{B}+i\underline{D}\right)\right)\right]\left(\underline{\theta}-i\underline{\varepsilon}\right).\end{gathered} \quad (10)$$

If to assume, that at dual turns $(\underline{\theta} - i\underline{\varepsilon})$ it is multiplied on $\exp[-i\theta]$ the constraint equation (10) supposes only the discrete dual transformations of a kind $\exp[i\pi/2]$ equivalent (5). Thus, there is remain in force E. Schrödinger's statement that there are only two electrodynamics invariant simultaneously concerning Lorenz's transformations and continuous dual turns - Maxwell electrodynamics and Born-Infeld electrodynamics.

The biquaternion formulation of the nonlinear equations of the non-commutative electrodynamics constructed here opens an opportunity for application of the advanced technique of a reduction to space-time of smaller dimensions [6, 7]. Thus significant interest represents research of a question on equivalence of nonlinearity of the systems obtained in this way and non-commutative coordinates in $\mathrm{R}^{(2.1)}$ and $\mathrm{R}^{(1.1)}$.

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