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Moyal dynamics of constraint systems

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Quantization of constraint systems within the Weyl-Wigner-Groenewold-Moyal framework is discussed. Constraint dynamics of classical and quantum systems is reformulated using the skew-gradient projection formalism. The quantum deformation of the Dirac bracket is generalized to match smoothly the classical Dirac bracket in and outside of the constraint submanifold in the limit $\hbar \to 0$.

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1. Introduction

Gauge symmetries provide mathematical basis for known fundamental interactions. Within the generalized Hamiltonian framework [1], gauge theories correspond to first-class constraints systems. Upon gauge fixing, these systems convert to second-class constraint systems. The operator quantization schemes for constraints systems have been developed by Dirac [1]. The path integral quantization has also been developed and found to be especially effective for gauge theories (for reviews see [2, 3]).

Besides conventional operator formulation of quantum mechanics and the path integral method, the popular approach to quantization of classical systems is based on the Groenewold star-product formalism [4]. It takes the origin from the Weyl's association rule [5] between operators in the Hilbert space and functions in phase space and the Wigner function [6]. The star-product formalism is known also under the names of the deformation quantization and the Moyal quantization [7, 8].

The skew-symmetric part of the star-product, named the Moyal bracket, governs the evolution of quantum systems in phase space, just like the Poisson bracket governs the evolution of classical unconstrained systems and the Dirac bracket governs the evolution of classical constraint systems. The Moyal bracket represents the quantum deformation of the Poisson bracket. The quantum deformation of the Dirac bracket has been constructed recently [9].

The outline of the paper is as follows: In the next Sect., we give a pedagogical introduction to the Weyl's association rule using the elegant method developed by Stratonovich [10] and give an introduction to the star-product formalism. More details on this subject can be found in articles [11–16].

The phase-space functions and the Dirac bracket do not make any physical sense outside of constraint submanifolds. In Ref. [9] we constructed the quantum deformation of the Dirac bracket on the constraint submanifold, sufficient for the purpose of generating time evolution of quantum constraint systems. It would, however, be interesting from the abstract point of view to have a quantum-mechanical extension of the Dirac bracket which matches smoothly at $\hbar \to 0$ with the classical Dirac bracket outside of the constraint submanifold also.

This problem is addressed and solved in Sects. III and IV. In Sect. III, we reformulate the classical constraint dynamics using projection formalism and present the classical Dirac bracket of functions in terms of the Poisson bracket of functions projected onto constraint submanifold.

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Table 1: Comparison of properties of Euclidean and symplectic spaces

Euclidean space $x, y \in \mathbb{R}^n$	**Symplectic space** $\xi, \zeta \in \mathbb{R}^{2n}$
Metric structure $g_{ij} = g_{ji}$ $g_{ij}g^{jk} = \delta_i^k$	Symplectic structure $I_{ij} = -I_{ji}$ $I_{ij}I^{jk} = \delta_i^k$
Scalar product $(x, y) = g_{ij}x^i y^j$	$\mathrm{Skew-scalar\ product}$ $(\xi, \zeta) = I_{ij}\xi^i\zeta^j$
Distance $L = \sqrt{(x-y, x-y)}$	Area $\mathcal{A} = (\xi, \zeta)$
Gradient $q(\bigtriangledown f)^i = g^{ij}\partial f/\partial x^j$	$\mathrm{Skew-gradient}$ $(Idf)^i \equiv -I^{ij}\partial f/\partial \xi^j$ $= \{\xi^i, f\}$
Scalar product of gradients of f and g $(\bigtriangledown f, \bigtriangledown g)$	Poisson bracket of f and g $(Idf, Idg) = \{f, g\}$
Orthogonality $g_{ij}x^i y^j = 0$	$\mathrm{Skew-orthogonality}$ $I_{ij}\xi^i\zeta^j = 0$

Sect. IV gives the quantum-mechanical generalization of the method proposed. Sects. III-D and IV-B,C contain new results, the others is a pedagogical exposition of earlier works (mainly [9]).

In Conclusion, we summarize results.

2. Weyl's association rule and the star-product

Systems with n degrees of freedom are described by $2n$ canonical coordinates and momenta $\xi^i = (q^1, ..., q^n, p_1, ..., p_n)$. These variables parameterize phase space $T_*\mathbb{R}^n$ defined as the cotangent bundle of n-dimensional configuration space $\mathbb{R}^n$. Canonical variables satisfy the Poisson bracket relations

$$\{\xi^k, \xi^l\} = -I^{kl}. \tag{2.1}$$

The skew-symmetric matrix I^{kl} has the form

$$\|I\| = \left\| \begin{matrix} 0 & -E_n \\ E_n & 0 \end{matrix} \right\| \tag{2.2}$$

where E_n is the $n \times n$ identity matrix and imparts to $T_*\mathbb{R}^n$ a skew-symmetric bilinear form. The phase space acquires thereby structure of symplectic space. The distance between two points in phase space is not defined. One can measure, however, areas stretched on any two vectors ξ^k and ζ^l as $\mathcal{A} = I_{kl}\xi^k\zeta^l$ where $I_{kl} = -I^{kl}$ so that $I_{kl}I^{lm} = \delta_k^m$.

Principal similarities and distinctions between Euclidean and symplectic spaces are cataloguized in Table 1. For skew-gradients of functions, short notation $Idf(\xi)$ is used.

In quantum mechanics, canonical variables ξ^i are associated to operators of canonical coordinates and momenta $\mathfrak{x}^i = (\mathfrak{q}^1, ..., \mathfrak{q}^n, \mathfrak{p}_1, ..., \mathfrak{p}_n)$ acting in the Hilbert space, which obey the commutation relations

$$[\mathfrak{x}^k, \mathfrak{x}^l] = -i\hbar I^{kl}. \tag{2.3}$$

The Weyl's association rule extends the correspondence $\xi^i \leftrightarrow \mathfrak{x}^i$ to phase-space functions $f(\xi) \in C^\infty(T_*\mathbb{R}^n)$ and operators $\mathfrak{f} \in Op(L^2(\mathbb{R}^n))$. It can be illustrated as follows:

$$\begin{aligned} \xi^i \in T_*\mathbb{R}^n &\longleftrightarrow \mathfrak{x}^i \in Op(L^2(\mathbb{R}^n)) \\ \{\xi^i, \xi^j\} &\longleftrightarrow -\frac{i}{\hbar}[\mathfrak{x}^i, \mathfrak{x}^j] \\ f(\xi) \in C^\infty(T_*\mathbb{R}^n) &\longleftrightarrow \mathfrak{f} \in Op(L^2(\mathbb{R}^n)) \end{aligned}$$

The set of operators $\mathfrak{f}$ acting in the Hilbert space is closed under multiplication of operators by c-numbers and summation of operators. Such a set constitutes vector space:

$$\left.\begin{array}{ccc} \left.\begin{array}{ccc} c \times f(\xi) & \longleftrightarrow & c\mathfrak{f} \\ f(\xi) + g(\xi) & \longleftrightarrow & \mathfrak{f} + \mathfrak{g} \end{array}\right\} \begin{array}{c}\text{vector}\\ \text{space}\end{array} \\ \begin{array}{ccc} f(\xi) \star g(\xi) & \longleftrightarrow & \mathfrak{f}\mathfrak{g} \end{array} \end{array}\right\} \text{algebra}$$

Elements of basis of such a vector space can be labelled by canonical variables ξ^i. The commonly used Weyl's basis looks like

$$\mathfrak{B}(\xi) \;=\; (2\pi\hbar)^n \delta^{2n}(\xi - \mathfrak{x}) = \int \frac{d^{2n}\eta}{(2\pi\hbar)^n} \exp(-\frac{i}{\hbar}\eta_k(\xi - \mathfrak{x})^k). \tag{2.4}$$

The objects $\mathfrak{B}(\xi)$ satisfy relations [9]

$$\begin{aligned} \mathfrak{B}(\xi)^+ &= \mathfrak{B}(\xi), \\ Tr[\mathfrak{B}(\xi)] &= 1, \\ \int \frac{d^{2n}\xi}{(2\pi\hbar)^n}\mathfrak{B}(\xi) &= \mathfrak{1}, \\ \int \frac{d^{2n}\xi}{(2\pi\hbar)^n}\mathfrak{B}(\xi)Tr[\mathfrak{B}(\xi)\mathfrak{f}] &= \mathfrak{f}, \\ Tr[\mathfrak{B}(\xi)\mathfrak{B}(\xi')] = (2\pi\hbar)^n\delta^{2n}(\xi &- \xi'), \\ \mathfrak{B}(\xi)\exp(-\frac{i\hbar}{2}\mathcal{P}_{\xi\xi'})\mathfrak{B}(\xi') &= (2\pi\hbar)^n\delta^{2n}(\xi - \xi')\mathfrak{B}(\xi'). \end{aligned}$$

Here,

$$\mathcal{P}_{\xi\xi'} = -I^{kl}\frac{\overleftarrow{\partial}}{\partial\xi^k}\frac{\overrightarrow{\partial}}{\partial\xi'^l}$$

is the so-called Poisson operator.

The Weyl's association rule for a function $f(\xi)$ and an operator $\mathfrak{f}$ has the form [10]

$$f(\xi) \;=\; Tr[\mathfrak{B}(\xi)\mathfrak{f}], \tag{2.5}$$

$$\mathfrak{f} \;=\; \int \frac{d^{2n}\xi}{(2\pi\hbar)^n} f(\xi)\mathfrak{B}(\xi). \tag{2.6}$$

In particular,

$$\xi^i \;=\; Tr[\mathfrak{B}(\xi)\mathfrak{x}^i] \tag{2.7}$$

$$\mathfrak{x}^i \;=\; \int \frac{d^{2n}\xi}{(2\pi\hbar)^n}\xi^i\mathfrak{B}(\xi). \tag{2.8}$$

The function $f(\xi)$ can be treated as the coordinate of $\mathfrak{f}$ in the basis $\mathfrak{B}(\xi)$, while the right side of Eq.(2.5) can be interpreted as the scalar product of $\mathfrak{B}(\xi)$ and $\mathfrak{f}$.

Alternative operator bases and their relations are discussed in Refs. [16, 17]. One can make, in particular, operator transforms on $\mathfrak{B}(\xi)$ and c-number transforms on ξ^i. Ambiguities in the choice of operator basis are connected to ambiguities in quantization of classical systems, better known as "operator ordering problem".

The set of operators is closed under multiplication of operators. The vector space of operators is endowed thereby with an associative algebra structure. Given two functions $f(\xi) = Tr[\mathfrak{B}(\xi)\mathfrak{f}]$ and $g(\xi) = Tr[\mathfrak{B}(\xi)\mathfrak{g}]$, one can construct a third function

$$f(\xi) \star g(\xi) = Tr[\mathfrak{B}(\xi)\mathfrak{f}\mathfrak{g}]. \tag{2.9}$$

This operation is called star-product. It has been introduced by Groenewold [4]. The explicit form of the star-product is as follows:

$$f(\xi) \star g(\xi) = f(\xi) \exp(\frac{i\hbar}{2}\mathcal{P})g(\xi), \tag{2.10}$$

where $\mathcal{P} = \mathcal{P}_{\xi\xi}$.

The star-product splits into symmetric and skew-symmetric parts

$$f \star g = f \circ g + \frac{i\hbar}{2} f \wedge g. \tag{2.11}$$

The skew-symmetric part $f \wedge g$ is known under the name of Moyal bracket. It is essentially unique [17]. It governs quantum evolution in phase space and endows the set of functions with the Poisson algebra structure:

$$\begin{array}{c} \text{physical observables} \\ \Updownarrow \\ \text{functions in phase space} \\ \Updownarrow \\ \overbrace{\underbrace{\underbrace{f+g,\ c\times f,}_{\text{vector space}}\ f\star g,}_{\text{algebra}}\ f\wedge g}^{\text{Poisson algebra}} \end{array} \tag{2.12}$$

The average values of a physical observable described by function $f(\xi)$ are calculated in terms of the Wigner function

$$W(\xi) = Tr[\mathfrak{B}(\xi)\mathfrak{r}]. \tag{2.13}$$

It is normalized to unity

$$\int \frac{d^{2n}\xi}{(2\pi\hbar)^n} W(\xi) = 1. \tag{2.14}$$

If $\mathfrak{f} \leftrightarrow f(\xi)$ and $\mathfrak{r} \leftrightarrow W(\xi)$ where $\mathfrak{r}$ is the density matrix, then

$$Tr[\mathfrak{f}\mathfrak{r}] = \int \frac{d^{2n}\xi}{(2\pi\hbar)^n} f(\xi) \star W(\xi) = \int \frac{d^{2n}\xi}{(2\pi\hbar)^n} f(\xi)W(\xi). \tag{2.15}$$

Under the sign of integral, the star-product can be replaced with the pointwise product [10].

Real functions in phase space stand for physical observables, which constitute in turn the Poisson algebra. If the associative product $f \star g$ does not commute, its skew-symmetric part gives automatically the skew-symmetric product which satisfies the Leibniz' law

$$f \wedge (g \star h) = (f \wedge g) \star h + g \star (f \wedge h). \tag{2.16}$$

This equation is valid separately for symmetric and skew-symmetric parts of the star-product. In the last case, Eq.(2.16) provides the Jacobi identity. The validity of the Leibniz' law allows to link the Moyal bracket with time derivative of functions and build up thereby an evolution equation for functions in phase space.

In classical limit, the Moyal bracket turns to the Poisson bracket:

$$\lim_{\hbar \to 0} f \wedge g = \{f, g\}.$$

3. Classical constraint systems in phase space

Second-class constraints $\mathcal{G}_a(\xi) = 0$ with $a = 1, ..., 2m$ and $m < n$ have the Poisson bracket relations which form a non-degenerate $2m \times 2m$ matrix

$$\det\{\mathcal{G}_a(\xi), \mathcal{G}_b(\xi)\} \neq 0. \tag{3.1}$$

If this condition is not fulfilled, it would mean that gauge degrees of freedom appear in the system. After imposing gauge-fixing conditions, we could arrive at inequality (3.1). Alternatively, breaking condition (3.1) would mean that constraint functions are dependent. After removing redundant constraints, we arrive at inequality (3.1).

Constraint functions are equivalent if they describe the same constraint submanifold. Within this class one can make transformations without changing dynamics.

3.1. Symplectic basis for constraint functions

For arbitrary point ξ of constraint submanifold $\Gamma^* = \{\xi : \mathcal{G}_a(\xi) = 0\}$, there is a neighborhood where one may find equivalent constraint functions in terms of which the Poisson bracket relations look like

$$\{\mathcal{G}_a(\xi), \mathcal{G}_b(\xi)\} = \mathcal{I}_{ab} \tag{3.2}$$

where

$$\mathcal{I}_{ab} = \left\| \begin{array}{ll} 0 & E_m \\ -E_m & 0 \end{array} \right\|. \tag{3.3}$$

Here, E_m is the identity $m \times m$ matrix, $\mathcal{I}_{ab}\mathcal{I}_{bc} = -\delta_{ac}$.

The global existence of symplectic basis (3.2) is an opened question in general case. The basis (3.2) always exists locally, i.e., in a finite neighborhood of any point of the constraint submanifold. This is sufficient for needs of perturbation theory. The formalism presented in this section can therefore to be used to formulate evolution problem of any second-class constraints system in phase space in the sense of the perturbation theory.

The existence of the local symplectic basis (3.2) is on the line with the Darboux's theorem (see, e.g., [18]) which states that in symplectic space around any point ξ there exists coordinate system in Δ_ξ such that $\xi \in \Delta_\xi$ where symplectic structure takes the standard canonical form. Symplectic spaces can be covered by such coordinate systems.

This is in contrast to Riemannian geometry where metric tensor at any given point x can always be made Minkowskian, but in any neighborhood of x the variance of the Riemannian metric with the Minkowskian metric is, in general, $\sim \Delta x^2$. Physically, by passing to inertial coordinate frame one can remove gravitation fields at any given point, but not in an entire neighborhood of that point. The Darboux's theorem states, reversely, that the symplectic structure can be made to take the standard canonical form in an entire neighborhood Δ_ξ of any point ξ. In Riemannian spaces, locally means at some given point. In symplectic spaces, locally means at some given point and in an entire neighborhood of that point.

Locally, all symplectic spaces are indistinguishable. Conditionally, one can say that any surface in symplectic space, including any constraint surface, is a plane.

In the view of this marked dissimilarity, the validity of Eqs.(2.1) in a finite domain looks indispensable.

3.2. Skew-gradient projection

The concept of skew-gradient projection $\xi_s(\xi)$ of canonical variables ξ onto constraint submanifold plays very important role in the Moyal quantization of constraint systems. Geometrically, skew-gradient projection acts along phase flows $Id\mathcal{G}^a(\xi)$ generated by constraint functions. These flows are commutative in virtue of Eqs.(3.2): Using Eqs.(3.2) and the Jacobi identity, one gets $\{\mathcal{G}^a, \{\mathcal{G}^b, f\}\} = \{\mathcal{G}^b, \{\mathcal{G}^a, f\}\}$ for any function f, so the point of intersection with Γ^* is unique. Skew-gradient projections are investigated in Refs. [19] and independently in Refs. [9, 20].

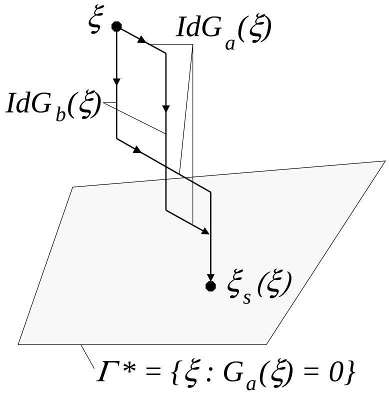

FIG. 1. Schematic presentation of skew-gradient projection onto constraint submanifold along commuting phase flows generated by constraint functions.

To construct skew-gradient projections, we start from equations

$$\{\xi_s(\xi), \mathcal{G}_a(\xi)\} = 0 \tag{3.4}$$

which say that point $\xi_s(\xi) \in \Gamma^*$ is left invariant by phase flows generated by $\mathcal{G}_a(\xi)$. Using symplectic basis (3.2) for the constraints and expanding

$$\xi_s(\xi) = \xi + X^a\mathcal{G}_a + \frac{1}{2}X^{ab}\mathcal{G}_a\mathcal{G}_b + ... \tag{3.5}$$

in the power series of $\mathcal{G}_a$, one gets

$$\xi_s(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!}\{...\{\{\xi, \mathcal{G}^{a_1}\}, \mathcal{G}^{a_2}\}, ...\mathcal{G}^{a_k}\}\mathcal{G}_{a_1}\mathcal{G}_{a_2}...\mathcal{G}_{a_k}. \tag{3.6}$$

Similar projection can be made for function $f(\xi)$:

$$f_s(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!}\{...\{\{f(\xi), \mathcal{G}^{a_1}\}, \mathcal{G}^{a_2}\}, ...\mathcal{G}^{a_k}\}\mathcal{G}_{a_1}\mathcal{G}_{a_2}...\mathcal{G}_{a_k}. \tag{3.7}$$

It satisfies

$$f_s(\xi) = f(\xi_s(\xi)). \tag{3.8}$$

Constraint functions are in involution with projected function:

$$\{f_s(\xi), \mathcal{G}_a(\xi)\} = 0. \tag{3.9}$$

Consequently, $f_s(\xi)$ does not vary along $Id\mathcal{G}_a(\xi)$, since

$$\{f(\xi), g(\xi)\} \equiv \frac{\partial f(\xi)}{\partial \xi^i}(Idg(\xi))^i.$$

Applying Eqs.(3.7) and (3.8) to constraint functions $\mathcal{G}_a(\xi)$, one concludes that the point $\xi_s(\xi)$ belongs to the constraint submanifold

$$\mathcal{G}_a(\xi_s(\xi)) = 0. \tag{3.10}$$

The constraint submanifold can therefore be described equivalently as $\Gamma^* = \{\xi_s(\xi) : \xi \in T_*\mathbb{R}^n\}$.

An average of function $f(\xi)$ is calculated using the probability density distribution $\rho(\xi)$ and the Liouville measure restricted to the constraint submanifold [21]:

$$< f >= \int \frac{d^{2n}\xi}{(2\pi)^n}(2\pi)^m \prod_{a=1}^{2m} \delta(\mathcal{G}_a(\xi)) f(\xi)\rho(\xi). \tag{3.11}$$

On the constraint submanifold $\xi_s(\xi) = \xi$, so $f(\xi)$ and $\rho(\xi)$ can be replaced with $f_s(\xi)$ and $\rho_s(\xi)$.

There exist therefore equivalence classes of functions in phase space:

$$f(\xi) \sim g(\xi) \leftrightarrow f_s(\xi) = g_s(\xi). \tag{3.12}$$

The symbol $\sim$ means that functions are equal in the weak sense, $f(\xi) \approx g(\xi)$, i.e., on the constraint submanifold. We shall see that symbols $\sim$ and $\approx$ acquire distinct meaning upon quantization. Note that $f(\xi) \sim f_s(\xi)$. Eqs.(3.8) and (3.10) imply $\mathcal{G}_a \sim 0$. Constraint functions belong to an equivalence class containing zero.

3.3. Dirac bracket in terms of Poisson bracket on constraint submanifold

Given hamiltonian function $\mathcal{H}$, the evolution of function f is described using the Dirac bracket [1]

$$\frac{\partial}{\partial t} f = \{f, \mathcal{H}\}_D. \tag{3.13}$$

In the symplectic basis (3.2), the Dirac bracket looks like

$$\{f, g\}_D = \{f, g\} + \{f, \mathcal{G}^a\}\{\mathcal{G}_a, g\}. \tag{3.14}$$

On the constraint submanifold, one has

$$\{f, g\}_D = \{f, g_s\} = \{f_s, g\} = \{f_s, g_s\}. \tag{3.15}$$

Calculation of the Dirac bracket can be replaced therefore with calculation of the Poisson bracket for functions projected onto the constraint submanifold.

Two functions are equivalent provided they coincide on the constraint submanifold. The hamiltonian functions determine the evolution of systems and play thereby special role. Two hamiltonian functions are equivalent if they generate within Γ^* phase flows whose projections onto the tangent plane of the constraint submanifold are identical. One may suppose that the equivalence relation for functions, defined above, does not apply to hamiltonian functions, since skew-gradients of hamiltonian functions enter the problem either. This is not the case, however. The components of the hamiltonian phase flow, which belong to a subspace spanned at Γ^* by phase flows of the constraint functions, do not affect dynamics and could be different, whereas the skew-gradient projection (3.7) *does not modify components of skew-gradients of functions, tangent to constraint submanifold.* We illustrate it schematically on Fig. 2. The geometrical sense of the Dirac bracket reduces to dropping the component of the hamiltonian phase flow which does not belong to tangent plane of the constraint submanifold. Equivalently, those components can be made to vanish with the help of the skew-gradient projection. $\mathcal{H}$ and $\mathcal{H}_s$ are thereby *dynamically equivalent*, so Eq.(3.12) characterizes an equivalence class for the hamiltonian functions either. Among functions of this class, $\mathcal{H}_s$ is the one whose phase flow is skew-orthogonal to phase flows of the constraint functions, i.e., $\{\mathcal{G}_a, \mathcal{H}_s\} = (Id\mathcal{G}_a, Id\mathcal{H}_s) = 0$.

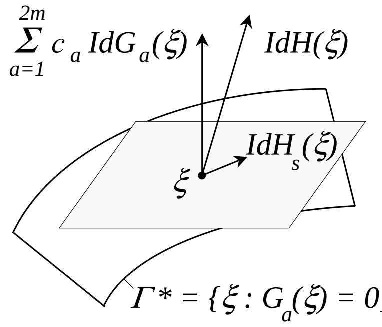

FIG. 2. Schematic presentation of phase flows $Id\mathcal{H}(\xi)$ and $Id\mathcal{H}_s(\xi)$ generated by hamiltonian function $\mathcal{H}(\xi)$ and projected hamiltonian function $\mathcal{H}_s(\xi)$ at point ξ of constraint submanifold Γ^*. The phase flow $Id\mathcal{H}_s(\xi)$ belongs to the tangent plane of Γ^*. The hamiltonian phase flow $Id\mathcal{H}(\xi)$ admits decomposition $Id\mathcal{H}(\xi) = \sum_{a=1}^{2m} c_a Id\mathcal{G}^a(\xi) + Id\mathcal{H}_s(\xi)$. Within the constraint submanifold (i.e. $\xi \in \Gamma^*$ and $\xi + d\xi \in \Gamma^*$) one has $d\mathcal{G}^a(\xi) = 0$ and therefore $0 = d\xi^i \partial\mathcal{G}^a(\xi)/\partial\xi^i = (Id\mathcal{G}^a(\xi), d\xi)$. The first term $\sum_{a=1}^{2m} c_a Id\mathcal{G}^a(\xi)$ is therefore skew-orthogonal to any vector $d\xi$ of the tangent plane.

Replacing $\mathcal{H}$ with $\mathcal{H}_s$, one can rewrite the evolution equation in terms of the Poisson bracket (cf. Eq.(3.13)):

$$\frac{\partial}{\partial t} f = \{f, \mathcal{H}_s\}. \tag{3.16}$$

The evolution does not mix up the equivalence classes.

The physical observables in second-class constraints systems are associated with the equivalence classes of real functions in the unconstrained phase space. The equivalence classes constitute a vector space $\mathcal{O}$ equipped with two multiplication operations, the associative pointwise product and the skew-symmetric Dirac bracket $\{,\}_D$, which confer $\mathcal{O}$ a Poisson algebra structure.

Instead of working with equivalence classes of functions $\mathcal{E}_f$, one can work with their representatives f_s defined uniquely by the skew-gradient projection. The one-to-one mapping $\mathcal{E}_f \leftrightarrow f_s$ induces a Poisson algebra structure on the set of projected functions. The sum $\mathcal{E}_f + \mathcal{E}_g$ converts to $f_s + g_s$, the associative product $\mathcal{E}_f\mathcal{E}_g$ converts to the pointwise product $f_s g_s$, while the Dirac bracket becomes the Poisson bracket:

$$\{f_s, g_s\}_D = \{f_s, g_s\}. \tag{3.17}$$

These operations satisfy the Leibniz' law and the Jacobi identity and, since $(f_s + g_s)_s = f_s + g_s$, $(f_s g_s)_s = f_s g_s$, and $\{f_s, h_s\}_s = \{f_s, h_s\}$, keep the set of projected functions closed.

3.4. Dirac bracket in terms of Poisson bracket on and outside of constraint submanifold

Outside of the constraint submanifold functions do not make any physical sense. It is sufficient thus to work with the Dirac bracket on the constraint submanifold. The evolution problem in such a case can consistently be formulated in terms of the Poisson bracket for functions projected onto the constraint submanifold.

The Dirac bracket is, however, well defined in the whole phase space. Redefinition of constraint functions by shifts $\mathcal{G}_a(\xi) \to \mathcal{G}_a(\xi)+$constant leaves the Dirac bracket unchanged, because it depends on derivatives of constraint functions only. It is not the case for the Poisson bracket applied to projected functions. This is why Eq.(3.15) is valid on constraint submanifold only.

One can modify projection formalism to fit the above-mentioned property of the Dirac bracket. Suppose we wish to find the Dirac bracket of functions $f(\zeta)$ and $g(\zeta)$ at a point $\zeta = \xi$ outside of the constraint submanifold. The intersection of level sets $\{\zeta : \mathcal{G}_a(\zeta) = \mathcal{G}_a(\xi)\}$ can be considered as new constraint submanifold defined by constraint functions

$$\Delta\mathcal{G}_a(\zeta) = \mathcal{G}_a(\zeta) - \mathcal{G}_a(\xi).$$

Projected functions depend thereby on both ζ and ξ:

$$f_S(\zeta) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}\{...\{\{f(\zeta), \Delta\mathcal{G}^{a_1}\}, \Delta\mathcal{G}^{a_2}\}, ...\Delta\mathcal{G}^{a_k}\}\Delta\mathcal{G}_{a_1}\Delta\mathcal{G}_{a_2}...\Delta\mathcal{G}_{a_k} \tag{3.18}$$

and similarly for $g(\zeta)$. The Poisson brackets are calculated with respect to ζ while ξ is a parameter. The appropriate extension looks like

$$\{f(\xi), g(\xi)\}_D \;=\; \{f(\zeta), g_S(\zeta)\}|_{\zeta=\xi} = \{f_S(\zeta), g(\zeta)\}|_{\zeta=\xi} = \{f_S(\zeta), g_S(\zeta)\}|_{\zeta=\xi}. \tag{3.19}$$

In Eq.(3.15) all four terms are pairwise distinct functions in the whole phase space. These functions coincide on the constraint submanifold only. In Eq.(3.19) all four terms coincide in the whole phase space. If $\xi \in \Gamma^*$, we reproduce the result (3.15) derived earlier.

4. Quantum constraint systems in phase space

Scheme presented in the previous Sect. 3 is suitable to approach description of quantum constraint systems in phase space. We give final results and refer to [9] for intermediate steps.

We remind that classical hamiltonian function $\mathcal{H}(\xi)$ and constraint functions $\mathcal{G}_a(\xi)$ are distinct in general from their quantum analogues $H(\xi)$ and $G_a(\xi)$. These dissimilarities are connected to the usual ambiguities in quantization of classical systems, being not specific for the problem we are interested in. It is required only

$$\lim_{\hbar\to 0} H(\xi) \;=\; \mathcal{H}(\xi),$$
$$\lim_{\hbar\to 0} G_a(\xi) \;=\; \mathcal{G}_a(\xi).$$

In what follows $\Gamma^* = \{\xi : G_a(\xi) = 0\}$.

4.1. Quantum deformation of the Dirac bracket on constraint submanifold

The quantum constraint functions $G_a(\xi)$ satisfy

$$G_a(\xi) \wedge G_b(\xi) = \mathcal{I}_{ab}. \tag{4.1}$$

In classical limit, $G_a(\xi)$ turn to $\mathcal{G}_a(\xi)$.

The quantum-mechanical version of the skew-gradient projections is defined with the use of the Moyal bracket

$$\xi_t(\xi) \wedge G_a(\xi) = 0. \tag{4.2}$$

Table 2. Brackets which govern evolution in phase space of functions (second column) and projected functions (third column) of classical systems (first row) and quantum systems (second row). The right upper corner shows the Dirac bracket expressed in terms of the Poisson bracket of functions projected onto the constraint submanifold. The left upper corner is the Poisson bracket. The left lower corner is the Moyal bracket, which represents the quantum deformation of the Poisson bracket. The operation $f_t \wedge g_t$ is the quantum deformation of the Dirac bracket.

Systems:	unconstrained	constrained
classical	$\{f, g\}$	$\{f_s, g_s\}$
quantum	$f \wedge g$	$f_t \wedge g_t$

The projected canonical variables have the form

$$\xi_t(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!}(...((\xi \wedge G^{a_1}) \wedge G^{a_2})... \wedge G^{a_k}) \circ G_{a_1} \circ G_{a_2} ... \circ G_{a_k}. \tag{4.3}$$

The quantum analogue of Eq.(3.7) is

$$f_t(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!}(...((f(\xi) \wedge G^{a_1}) \wedge G^{a_2})... \wedge G^{a_k}) \circ G_{a_1} \circ G_{a_2} ... \circ G_{a_k}. \tag{4.4}$$

The function $f_t(\xi)$ obeys equation

$$f_t(\xi) \wedge G_a(\xi) = 0. \tag{4.5}$$

The evolution equation which is the analogue of Eq.(3.16) takes the form

$$\frac{\partial}{\partial t} f(\xi) = f(\xi) \wedge H_t(\xi) \tag{4.6}$$

where $H_t(\xi)$ is the hamiltonian function projected onto the constraint submanifold as prescribed by Eq.(4.4). Taking projection of Eq.(4.6) we get evolution equation in the closed form for projected functions:

$$\frac{\partial}{\partial t} f_t(\xi) = f_t(\xi) \wedge H_t(\xi) \tag{4.7}$$

The quantum deformation of the Dirac bracket represents the Moyal bracket for two functions projected quantum-mechanically onto the constraint submanifold.

The formal structure of the dynamical quantum system is described by the scheme (2.12) with the word "functions" replaced by the phrase "projected functions" and f and g replaced by f_t and g_t, respectively. The star-product is an associative operation, whereas the Moyal bracket for projected functions satisfies the Leibniz' law and, respectively, the Jacobi identity.

Projected functions in phase space are objects associated to quantum observables. Functions which have the same projections are physically equivalent. We can unify such functions into equivalence classes. The star-product and the Moyal bracket for projected functions generate for equivalence classes a Poisson algebra structure accordingly.

The bracket $f_t \wedge g_t$ constructed in [9] gives the deformation of the Dirac bracket on Γ^*. What about the whole phase space?

4.2. Quantum deformation of the Dirac bracket on and outside of constraint submanifold

One can generalize the operation $f_t \wedge g_t$ to match in classical limit the Dirac bracket outside of the constraint submanifold. We can proceed like in the classical case by writing projected functions in the form

$$f_T(\zeta) \ = \ \sum_{k=0}^{\infty} \frac{1}{k!}(...((f(\zeta) \wedge \Delta G^{a_1}) \wedge \Delta G^{a_2}) \wedge ...\Delta G^{a_k}) \circ \Delta G_{a_1} \circ \Delta G_{a_2} ... \circ \Delta G_{a_k} \quad (4.8)$$

where

$$\Delta G_a(\zeta) = G_a(\zeta) - G_a(\xi).$$

The Moyal brackets and the $\circ$-products entering this equation are calculated with respect to ζ. The desired extension looks like

$$f_T(\zeta) \wedge g_T(\zeta)|_{\zeta=\xi}. \quad (4.9)$$

It is assumed that the constraint functions $G_a(\xi)$ satisfy the bracket relations (4.1) at $\xi \notin \Gamma^*$. Expression (4.9) is valid on and outside of the constraint submanifold. If $\xi \in \Gamma^*$, we reproduce operation $f_t(\xi) \wedge g_t(\xi)$ announced earlier.

4.3. Completeness of the set of projected operators of canonical coordinates and momenta

The set of operators $\mathfrak{x}^i$ is known to be complete, so that any operator $\mathfrak{f}$ can be represented as a symmetrized (probably infinite) weighted sum of products of operators $\mathfrak{x}^i$. In the sense of the Taylor expansion, one can write $\mathfrak{f} = f(\mathfrak{x})$. The one-to-one correspondence between operators $\mathfrak{f} \in Op(L^2(\mathbb{R}^n))$ and functions in phase space $f(\xi)$, based on the Taylor expansion, is equivalent to the Weyl's association rule.

The similar completeness condition holds for projected operators of canonical variables $\mathfrak{x}_t^i$ which are inverse Weyl's transforms of $\xi_t^i(\xi)$. Apparently, any operator $\mathfrak{f}$ acting in the Hilbert space can be represented as an operator function $\varphi(\mathfrak{G}^a, \mathfrak{x}_t^i)$. Applying projection to the symmetrized product of k constraint operators $\mathfrak{G}^a$, which are inverse Weyl's transforms of $G^a(\xi)$, one gets a series like $1 - k + \frac{1}{2!}k(k-1) + \ldots = (1-1)^k = 0$, and so

$$(\mathfrak{G}^{(a_1}\mathfrak{G}^{a_2}...\mathfrak{G}^{a_k)})_t = 0. \quad (4.10)$$

The Taylor series of $\varphi(\mathfrak{G}^a, \mathfrak{x}_t^i)$ generates thereby vanishing terms involving $\mathfrak{G}^a$. We thus obtain

$$(\varphi(\mathfrak{G}^a, \mathfrak{x}_t^i))_t = \varphi(0, \mathfrak{x}_t^i). \quad (4.11)$$

Respectively, any function projected quantum-mechanically onto the constraint submanifold can be represented in the form

$$f_t(\xi) = \varphi(\star\xi_t(\xi)). \quad (4.12)$$

One can pass to classical limit to get Eq.(3.8). Constructing $\varphi(\xi)$ from $f(\xi)$ is a non-trivial task equivalent to solving constraints. The operator counterpart of Eq.(4.12),

$$\mathfrak{f}_t = \varphi(\mathfrak{x}_t), \quad (4.13)$$

demonstrates the completeness of projected set of operators of canonical coordinates and momenta. Accordingly, Eq.(4.12) shows completeness of the set of $\xi_t^i(\xi)$ in description of projected functions. It is worthwhile to notice that Eq.(4.10) does not extend to antisymmetric products of $\mathfrak{G}^a$ as one sees from $[\mathfrak{G}^a, \mathfrak{G}^b]_t = (-\mathcal{I}^{ab})_t = -\mathcal{I}^{ab} \neq [\mathfrak{G}_t^a, \mathfrak{G}_t^b] = 0$ where condition $\mathfrak{G}_t^a = 0$ is taken into account.

5. Conclusion

We made short introduction to the Weyl's association rule and the Groenewold star-product technique for unconstrained and constraint systems. The attention was focused to the evolution problem.

A generalization of the quantum deformation of the Dirac bracket is constructed to match smoothly classical Dirac bracket in the whole phase space at $\hbar \to 0$.

The use of skew-gradient projection formalism allows to treat unconstrained and constraint systems essentially on the same footing. Projections of solutions of quantum evolution equations onto the constraint submanifold comprise the entire information on quantum dynamics of constraint systems.

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