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The wave equation for a scalar particle in Riemannian space, non-minimal interaction and non-relativistic approximation

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The procedure of the non-relativistic approximation in the theory of scalar particle, charged and neutral, is investigated in the background of Riemannian space-time. A generalized covariant Schrödinger equation is derived when taking into account non-minimal interaction term through scalar curvature $R(x)$, it substantially differs from the conventional generally covariant Schrödinger equation produced when $R(x) = 0$. It is shown that the the non-relativistic wave function is always complex-valued irrespective of the type of relativistic scalar particle, charged or neutral, taken initially. The theory of vector particle proves the same property: even if the wave function of the relativistic particle of spin 1 is taken real, the corresponding wave function in the non-relativistic approximation is complex-valued.

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1. Introduction

The wave equation for a scalar particle in curved space-time with the metric

$$dS^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

is taken in the form [1,2]

$$[\ (i\hbar\, \nabla_\alpha + \frac{e}{c} A_\alpha)\ g^{\alpha\beta}(x)\ (i\hbar\, \nabla_\beta + \frac{e}{c} A_\beta)\ - \frac{\hbar^2}{6}\ R\ - m^2 c^2\ ]\ \Psi(x) = 0\ . \qquad (1)$$

Take notice on additional interaction term through scalar curvature $R(x)$. In the following, to abridge formulas the notation will be used: $e/\hbar c \Longrightarrow e$, $mc/h \Longrightarrow m$, then the main equation reads

$$[\ (i\ \nabla_\alpha + e\ A_\alpha)\ g^{\alpha\beta}(x)\ (i\ \nabla_\beta + e\ A_\beta)\ - \frac{1}{6}\ R\ - m^2\ ]\ \Psi(x) = 0\ . \qquad (2)$$

This equation may be changed to the form more convenient in application. For this end, let it be rewritten as

$$\begin{aligned} \big[\ i^2\ \nabla_\alpha g^{\alpha\beta}(x)\nabla_\beta + ie\ (\nabla_\alpha g^{\alpha\beta}(x) A_\beta) + 2ie\ A_\alpha g^{\alpha\beta}(x)\nabla_\beta + \\ + e^2\ A_\alpha g^{\alpha\beta}(x)\ A_\beta - R/6\ - m^2 \big]\ \Psi(x) = 0\ . \end{aligned} \qquad (3)$$

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With the use of the known relations [3]

$$\nabla_\alpha g^{\alpha\beta}(x) \nabla_\beta \, \Phi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} \; \Psi \; ,$$

$$\nabla_\alpha g^{\alpha\beta} A_\beta = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} g^{\alpha\beta} A_\beta \; , \qquad g = \det \, (g_{\alpha\beta}) \tag{4}$$

eq. (3) is changed to

$$\Big[\; i^2 \; \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} g^{\alpha\beta}(x) \frac{\partial}{\partial x^\beta} + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} g^{\alpha\beta}(x) A_\beta +$$

$$+ 2ie \; A_\alpha g^{\alpha\beta} \frac{\partial}{\partial_\beta} + e^2 \; A_\alpha g^{\alpha\beta}(x) \; A_\beta - \frac{1}{6} \; R \; - m^2 \Big] \; \Psi(x) = 0 \; , \tag{5}$$

or

$$\Big[\; \frac{1}{\sqrt{-g}} (i \frac{\partial}{\partial x^\alpha} + eA_\alpha) \; \sqrt{-g} g^{\alpha\beta}(x) (i \frac{\partial}{\partial x^\beta} + eA_\beta) - \frac{1}{6} \; R \; - m^2 \Big] \; \Psi(x) = 0 \; . \tag{6}$$

In ordinary units it reads as

$$\Big[\; \frac{1}{\sqrt{-g}} (i\hbar \frac{\partial}{\partial x^\alpha} + \frac{e}{c} A_\alpha) \; \sqrt{-g} g^{\alpha\beta}(x) (i\hbar \frac{\partial}{\partial x^\beta} + \frac{e}{c} A_\beta) - \frac{\hbar^2}{6} \; R \; - m^2 c^2 \Big] \; \Psi(x) = 0 \; . \tag{7}$$

What is the Schrödinger's non-relativistic equation in the curved space-time? It might exist only in space-time models with special metric (see treatment of the non-relativistic limit for particles with spin 1/2 in [4,5] and spin 1 in [6])

$$dS^2 = c^2 dt^2 + g_{kl}(x) \; dx^k dx^l \tag{8}$$

One might expect to reach clarity if one could follow details of the limiting procedure from generally covariant Klein-Fock equation to Schrödinger one. This is the goal of the present paper.

2. Non-relativistic approximation in the curved space-time

Let us begin with a generally covariant first order equations for a scalar particle (take notice to the additional interaction term through the Ricci scalar [1,2]):

$$(i \; \nabla_\alpha + \frac{e}{c\hbar} A_\alpha) \; \Phi = \frac{mc}{\hbar} \; \Phi_\alpha \; ,$$

$$(i \; \nabla_\alpha + \frac{e}{c\hbar} A_\alpha) \; \Phi^\alpha = \frac{mc}{\hbar} (1 + \sigma \; \frac{R(x)}{m^2 c^2 / \hbar^2}) \; \Phi \; , \tag{9}$$

For brevity the will be used

$$1 + \sigma \; \frac{R(x)}{m^2 c^2 / \hbar^2} = \Gamma(x) \; .$$

Eq. (9) reads

$$(i \; \partial_\alpha + \frac{e}{c\hbar} A_\alpha) \; \Phi(x) = \frac{mc}{\hbar} \; \Phi_\alpha \; ,$$

$$(\frac{i}{\sqrt{-g}} \; \frac{\partial}{\partial x^\alpha} \sqrt{-g} + \frac{e}{c\hbar} \; A_\alpha) \; g^{\alpha\beta} \Phi_\beta = \frac{mc}{\hbar} \; \Gamma \; \Phi \; . \tag{10}$$

In the space-time models of the type (12), one can easily separate time- and space- variables in eq. (10):

$$
\begin{aligned}
(i\,\partial_0 + \frac{e}{c\hbar} A_0)\;\Phi &= \frac{mc}{\hbar}\;\Phi_0\;, \\
(i\,\partial_l + \frac{e}{c\hbar} A_l)\;\Phi(x) &= \frac{mc}{\hbar}\;\Phi_l\;, \\
(\frac{i}{\sqrt{-g}}\;\frac{\partial}{\partial x^0}\sqrt{-g} + \frac{e}{c\hbar}\;A_0)\;\Phi_0 &+ \\
+(\frac{i}{\sqrt{-g}}\;\frac{\partial}{\partial x^k}\sqrt{-g} + \frac{e}{c\hbar}\;A_k)\;g^{kl}\Phi_l &= \frac{mc}{\hbar}\;\Gamma\;\Phi\;,
\end{aligned}
$$

or

$$
\begin{aligned}
(i\,\partial_0 + \frac{e}{c\hbar} A_0)\;\Phi &= \frac{mc}{\hbar}\;\Phi_0\;, \\
(i\,\partial_l + \frac{e}{c\hbar} A_l)\;\Phi &= \frac{mc}{\hbar}\;\Phi_l\;, \\
(i\frac{\partial}{\partial x^0} + \frac{i}{\sqrt{-g}}\;\frac{\partial\sqrt{-g}}{\partial x^0} + \frac{e}{c\hbar}\;A_0)\;\Phi_0 &+ \\
+(\frac{i}{\sqrt{-g}}\;\frac{\partial}{\partial x^k}\sqrt{-g} + \frac{e}{c\hbar}\;A_k)\;g^{kl}\Phi_l &= \frac{mc}{\hbar}\;\Gamma\;\Phi\;. \qquad (11)
\end{aligned}
$$

Now one should separate the rest energy - term by means of the substitutions:

$$
\Phi \Longrightarrow exp\,[-i\frac{mc^2 t}{\hbar}]\;\;\Phi\;, \qquad \Phi_0 \Longrightarrow exp\,[-i\frac{mc^2 t}{\hbar}]\;\;\Phi_0\;, \qquad \Phi_l \Longrightarrow exp\,[-i\frac{mc^2 t}{\hbar}]\;\;\Phi_l\;.
$$

As a result, eq. (11) will give

$$
\begin{aligned}
(\frac{i}{c}\;\partial_t + \frac{mc}{\hbar}\;\; + \frac{e}{c\hbar} A_0)\;\Phi(x) &= \frac{mc}{\hbar}\;\Phi_0(x)\;, \\
(\frac{i}{c}\;\partial_t + \frac{mc}{\hbar} + \frac{i}{\sqrt{-g}}\;\frac{\partial\sqrt{-g}}{\partial} + \frac{e}{c\hbar}\;A_0)\;\Phi_0 &+ \\
+(\frac{i}{\sqrt{-g}}\;\frac{\partial}{\partial x^k}\sqrt{-g} + \frac{e}{c\hbar}\;A_k)\;g^{kl}\Phi_l &= \frac{mc}{\hbar}\;\Gamma\;\Phi(x)\;, \\
(i\,\partial_l + \frac{e}{c\hbar} A_l)\;\Phi(x) &= \frac{mc}{\hbar}\;\Phi_l(x)\;,
\end{aligned}
$$

or

$$
\begin{aligned}
(i\hbar\,\partial_t + mc^2\;\; + eA_0)\;\Phi(x) &= mc^2\;\Phi_0(x)\;, \qquad (12) \\
(i\hbar\,\partial_t + mc^2 + i\hbar\frac{1}{\sqrt{-g}}\;\frac{\partial\sqrt{-g}}{\partial t} + e\;A_0)\;\Phi_0 &+ \\
+c(\frac{i\hbar}{\sqrt{-g}}\;\frac{\partial}{\partial x^k}\sqrt{-g} + \frac{e}{c}\;A_k)\;g^{kl}\Phi_l &= mc^2\;\Gamma\;\Phi(x)\;, \qquad (13) \\
(i\,\hbar\,\partial_l + \frac{e}{c} A_l)\;\Phi(x) &= mc\;\Phi_l(x)\;. \qquad (14)
\end{aligned}
$$

With the help of (14), the non-dynamical variable Φ_l can be readily excluded:

$$
\begin{aligned}
(i\hbar\,\partial_t + mc^2\;\; + eA_0)\;\Phi(x) &= mc^2\;\Phi_0(x)\;, \qquad (15) \\
(\;i\hbar\,\partial_t + mc^2 + i\hbar\frac{1}{\sqrt{-g}}\;\frac{\partial\sqrt{-g}}{\partial t} + e\;A_0\;)\;\Phi_0 &+ \\
+\frac{1}{m}\;[\;(\frac{i\hbar}{\sqrt{-g}}\;\partial_k\sqrt{-g} + \frac{e}{c}\;A_k)\;g^{kl}(i\,\hbar\,\partial_l + \frac{e}{c}A_l)\;]\;\Phi(x) &= mc^2\;\Gamma\;\Phi(x)\;. \qquad (16)
\end{aligned}
$$

Now we are to introduce a small φ and big Ψ components:

$$\Phi - \Phi_0 = \varphi, \qquad \Phi + \Phi_0 = \Psi \; ; \tag{17}$$

$$\Phi = \frac{\Psi + \varphi}{2}, \qquad \Phi_0 = \frac{\Psi - \varphi}{2} \; . \tag{18}$$

Substituting eq. (18)into (15) and (16) one gets

$$(i\hbar\,\partial_t + mc^2 \; + eA_0)\,\frac{\Psi + \varphi}{2} = mc^2\,\frac{\Psi - \varphi}{2}\;, \tag{19}$$

$$(\, i\hbar\,\partial_t + mc^2 + i\hbar\frac{1}{\sqrt{-g}}\,\frac{\partial\sqrt{-g}}{\partial t} + e\,A_0 \,)\,\frac{\Psi - \varphi}{2} +$$

$$+\frac{1}{m}\,[\,(\frac{i\hbar}{\sqrt{-g}}\,\partial_k\sqrt{-g} + \frac{e}{c}\,A_k)\,g^{kl}(i\,\hbar\,\partial_l + \frac{e}{c}A_l)\,]\,\frac{\Psi + \varphi}{2} = mc^2\,\Gamma\,\frac{\Psi + \varphi}{2}\;. \tag{20}$$

or after simple calculation we arrive at

$$(i\hbar\,\partial_t + eA_0)\,\frac{+\varphi + \Psi}{2} = -mc^2\,\varphi\;, \tag{21}$$

$$(\, i\hbar\,\partial_t + i\hbar\frac{1}{\sqrt{-g}}\,\frac{\partial\sqrt{-g}}{\partial t} + e\,A_0 \,)\,\frac{\Psi - \varphi}{2} +$$

$$+\frac{1}{m}\,[\,(\frac{i\hbar}{\sqrt{-g}}\,\partial_k\sqrt{-g} + \frac{e}{c}\,A_k)\,g^{kl}(i\,\hbar\,\partial_l + \frac{e}{c}A_l)\,]\,\frac{\Psi + \varphi}{2} =$$

$$= mc^2\,(\Gamma + 1)\,\frac{\varphi}{2} + mc^2\,(\Gamma - 1)\,\frac{\Psi}{2}\;. \tag{22}$$

In this point, it is better to consider two different cases.

The first possibility is when one poses an additional requirement $\Gamma = 1$, which means the absence of the non-minimal interaction term through R-scalar. Then at $\Gamma = 1$, from the previous equations – ignoring small component compared with big one – it follows

$$(i\hbar\,\partial_t + eA_0)\,\frac{\Psi}{2} = -mc^2\,\varphi\;, \tag{23}$$

$$(\, i\hbar\,\partial_t + i\hbar\frac{1}{\sqrt{-g}}\,\frac{\partial\sqrt{-g}}{\partial t} + e\,A_0 \,)\,\frac{\Psi}{2} +$$

$$+\frac{1}{m}\,[\,(\frac{i\hbar}{\sqrt{-g}}\,\partial_k\sqrt{-g} + \frac{e}{c}\,A_k)\,g^{kl}(i\,\hbar\,\partial_l + \frac{e}{c}A_l)\,]\,\frac{\Psi}{2} = mc^2\,\varphi\;. \tag{24}$$

Finally, excluding the small constituent we arrive at

$$[\,i\hbar\,(\partial_t + \frac{1}{2\sqrt{-g}}\,\frac{\partial\sqrt{-g}}{\partial t}) + e\,A_0\,]\,\Psi =$$

$$= \frac{1}{2m}\,[\,(\frac{i\hbar}{\sqrt{-g}}\,\partial_k\sqrt{-g} + \frac{e}{c}\,A_k)\,(-g^{kl})\,(i\,\hbar\,\partial_l + \frac{e}{c}A_l)\,]\,\Psi \tag{25}$$

With the help of substitution $\Psi \implies (-g)^{-1/4}\,\Psi$ the obtained equation can be simplified:

$$(\, i\hbar\,\partial_t + e\,A_0 \,)\,\Psi =$$

$$= \frac{1}{2m}\,[\,(\frac{i\hbar}{\sqrt{-g}}\,\partial_k\sqrt{-g} + \frac{e}{c}\,A_k)\,(-g^{kl})\,(i\,\hbar\,\partial_l + \frac{e}{c}A_l)\,]\,\Psi\;, \tag{26}$$

which is the the Schrödinger equation in curved space.

The second possibility when $\Gamma \neq 1$, then from (22)

$$(i\hbar\, \partial_t + eA_0)\, \frac{\Psi}{2} = -mc^2\, \varphi\,, \tag{27}$$

$$\begin{gathered}(\, i\hbar\, \partial_t + i\hbar \frac{1}{\sqrt{-g}}\, \frac{\partial \sqrt{-g}}{\partial t} + e\, A_0\,)\, \frac{\Psi}{2} + \\ + \frac{1}{m}\, [\, (\frac{i\hbar}{\sqrt{-g}}\, \partial_k \sqrt{-g} + \frac{e}{c}\, A_k)\, g^{kl} (i\, \hbar\, \partial_l + \frac{e}{c} A_l)\,]\, \frac{\Psi}{2} = \\ = mc^2\, (\Gamma + 1)\, \frac{\varphi}{2} + mc^2\, (\Gamma - 1)\, \frac{\Psi}{2}\,. \end{gathered} \tag{28}$$

With the use of (27) we can derive the following equation for the big component Ψ:

$$\begin{gathered}(\, i\hbar\, \partial_t + i\hbar \frac{1}{\sqrt{-g}}\, \frac{\partial \sqrt{-g}}{\partial t} + e\, A_0\,)\, \frac{\Psi}{2} + \\ + \frac{(\Gamma + 1)}{2}\, (i\hbar\, \partial_t + eA_0)\, \frac{\Psi}{2}\, - mc^2\, (\Gamma - 1)\, \frac{\Psi}{2} = \\ - \frac{1}{2m}\, [\, (\frac{i\hbar}{\sqrt{-g}}\, \partial_k \sqrt{-g} + \frac{e}{c}\, A_k)\, g^{kl} (i\, \hbar\, \partial_l + \frac{e}{c} A_l)\,]\, \Psi. \end{gathered} \tag{29}$$

This equation can be rewritten as follows:

$$\begin{gathered}[\, (\frac{1}{2} + \frac{1}{2}\, \frac{\Gamma(x) + 1}{2}) (i\hbar \partial_t + e\, A_0) + \frac{i\hbar}{2\sqrt{-g}}\, \frac{\partial \sqrt{-g}}{\partial t})\,]\, \Psi = \\ = \frac{1}{2m}\, [\, (\frac{i\hbar}{\sqrt{-g}}\, \partial_k \sqrt{-g} + \frac{e}{c}\, A_k)\, (-g^{kl})\, (i\, \hbar\, \partial_l + \frac{e}{c} A_l)\,]\, \Psi + mc^2\, \frac{(\Gamma(x) - 1)}{2}\, \Psi \end{gathered} \tag{30}$$

It remains to recall that

$$\Gamma(x) = 1 + \frac{1}{6}\, \frac{\hbar^2 R(x)}{m^2 c^2}\,,$$

so the previous equation will take the form

$$\begin{gathered}\left[\, (1 + \frac{1}{24}\, \frac{\hbar^2 R(x)}{m^2 c^2})\, (i\hbar \partial_t + e\, A_0) + \frac{i\hbar}{2\sqrt{-g}}\, \frac{\partial \sqrt{-g}}{\partial t})\, \right]\, \Psi = \\ = \frac{1}{2m}\, [\, (\frac{i\hbar}{\sqrt{-g}}\, \partial_k \sqrt{-g} + \frac{e}{c}\, A_k)\, (-g^{kl})\, (i\, \hbar\, \partial_l + \frac{e}{c} A_l)\,]\, \Psi + mc^2\, \frac{\hbar^2 R}{12 m^2 c^2}\, \Psi \end{gathered} \tag{31}$$

and finally

$$\begin{gathered}\left[\, (1 + \frac{1}{24}\, \frac{\hbar^2 R(x)}{m^2 c^2})\, (i\hbar \partial_t + e\, A_0) + \frac{i\hbar}{2\sqrt{-g}}\, \frac{\partial \sqrt{-g}}{\partial t}\, \right]\, \Psi = \\ = \frac{1}{2m}\, \left[\, (\frac{i\hbar}{\sqrt{-g}}\, \partial_k \sqrt{-g} + \frac{e}{c}\, A_k)\, (-g^{kl})\, (i\, \hbar\, \partial_l + \frac{e}{c} A_l)\,]\, + \hbar^2 \frac{R}{6} \right]\, \Psi \end{gathered} \tag{32}$$

which should be considered as a Schrödinger equation in a space-time with non-vanishing scalar curvature $R(x) \neq 0$ when allowing for a non-minimal interaction term through scalar curvature $R(x)$.

3. Conclusions

In addition, several general comments may be given. The wave function of Schrödinger equation Ψ does not coincide with the initial scalar Klei-Fock Φ. Instead we have the following

$$\Psi = \Phi + \Phi_0, \qquad \Phi_0 \text{ belongs to } \{ \Phi_0, \Phi_1, \Phi_2, \Phi_3 \} . \tag{33}$$

One may have looked at this fact as a non-occasional an even necessary one. Indeed, let one start with a neutral scalar particle theory. Such a particle cannot interact with electromagnetic field and its wave function is real. However, by general consideration, certain non-relativistic limit in this theory must exist. It is the case in fact: the added term in (33)

$$\Phi_0 = i \frac{\hbar}{mc} \nabla_0 \Phi \tag{34}$$

is imaginary even if $\Phi^* = +\Phi$. All the more, that situation is in accordance with the the mathematical structure of the Schrödinger equation itself, it cannot be written for real wave function at all.

One other argument can be added. The same property can be seen in the theory of a vector particle: even if the wave function of the relativistic particle of spin 1 is taken real, the corresponding wave function in the non-relativistic approximation turn to be complex-valued (see [6]):

$$\psi(x) = \frac{1}{2} [\Phi_i(x) + i \, E_i(x)] , \qquad E_i(x) = \Phi_{0i}(x) . \tag{35}$$

By general consideration, one may expect an analogous result in the theory of a spin 1/2 particle: if the non relativistic approximation is done in the theory of Majorana neutral particle with the real 4-spinor wave function then the corresponding Pauli 2-spinor must be complex-valued.

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