Nonlinear Dynamics and Applications. Vol. 13 (2006) 270 - 274

Functional integrals on the space of functions of multiple variables and the measure interpolation formula

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The formula of measure interpolation was used for constructing of approximation formulas for functional integrals determined on spaces of functions of many variables. Considered in this paper approximations is specific for such spaces and can not be given from formulas which were constructed for generalized approximations of integrals in linear spaces.

PACS numbers: 02.30.Cj, 02.30.Mv, 02.60.-x, 31.15.Kb

Keywords: functional integral, measure interpolation, approximation, functions of many variables

1. Introduction

In this paper we consider functional integrals with respect to Gaussian measure on the space $X_m \equiv X_m(\mathbf{T})$, $\mathbf{T} = \prod_{\alpha=1}^{m} [0, T_\alpha]$, $T_\alpha > 0$, $\alpha = 1, 2, \ldots, m$, - real numbers. We suppose, that the mean value of a measure equals to zero. For calculation of this integrals we shall build approximations based on interpolation of correlation function of the measure. Formulas based on measure interpolation for case of spaces of functions of single variable were constructed in papers [1, 2]. Considered in this paper approximations are specific for spaces of functions of many variables and can not be given from formulas, received for general case of approximations of integrals in linear space. In particular, constructed in this paper approximations can be used for calculation of functionals of the view

$$f(x) = \exp\left\{ \int_{\mathbf{T}} V(x(t))dt \right\} \equiv \exp\{V_{\mathbf{T}}\}, \qquad (1),$$

where $x \in X_m(\mathbf{T})$, $V(u) \in R$.

2. Formula with derivatives of integrated functional

We need to remind the formula of measure interpolation (see e.g. [3])

$$I \equiv \int_X F(x(\cdot))d\mu(x) = \int_X F(x(\cdot))d\mu_0(x) + \frac{1}{2}\int_0^1 \int_X \int_0^T \int_0^T (B-B_0)(s_1, s_2)F^{(2)}(x; s_1, s_2)\, ds_1 ds_2\, d\mu_u(x)du,$$

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which we use for constructing of approximation formulas.

The expression $V_{\mathbf{T}} = \int\limits_{\mathbf{T}} V(x(t))dt$, in exponent of the expression (1) can be represented in the form

$$V_{\mathbf{T}} = \sum_{p_1,\dots,p_m=0}^{1} \int\limits_{T_1^{p_1}} \cdots \int\limits_{T_m^{p_m}} V(x(t_1,\dots,t_m))dt_1\cdots dt_m \equiv \sum_{\mathbf{p}} \int\limits_{\mathbf{T^p}} V(x(t))dt, \qquad (2)$$

where $T_\alpha^{(0)} = [0, T_\alpha/2]$, $T_\alpha^1 = [T_\alpha/2, T_\alpha]$, $\alpha = 1,\dots,m$; $\mathbf{p} = (p_1,\dots,p_m)$, $\mathbf{T^p} = T_1^{(p_1)} \times \dots \times T_m^{p_m}$ $\sum\limits_{\mathbf{p}} = \sum\limits_{p_1,\dots,p_m=0}^{1}$. After substituting (2) in (1) function f has the form

$$f(x) = \prod_{\mathbf{p}} \exp\Big\{ \int\limits_{\mathbf{T^p}} V(x(t))dt \Big\}, \qquad (3).$$

In this form multipliers are functionals, which not equals to zero on non intercepted sets. According to general idea for given Gaussian measure μ with correlation functional $B(t,s)$ we shall determine Gaussian measure μ_0 on $X_m(\mathbf{T})$ by the next rule: an integral for functionals of the form (1) with respect to this measure must be equal to production of integrals of co-multipliers in (3) on spaces of functions $X_m(\mathbf{T})$.

Let us consider Gaussian measure $\mu_{s_1,\dots,s_m}$ on $X_m(\mathbf{T})$ with the correlation function

$$B_{s_1,\dots,s_m}(t,\tau) = \sum_{\Gamma} \prod_{\alpha\in\Gamma} s_\alpha \prod_{\alpha\in\Gamma^c} (1-s_\alpha) B_{\Gamma^c}(t,\tau), \qquad (4)$$

where Γ — subset of the set $\{1,\dots,m\}$; in particular, it can be empty set or set $\{1,\dots,m\}$; Γ^c — addition of the set Γ in $\{1,\dots,m\}$; $B_{\Gamma^c}(t,\tau) = B(t,\tau)$ on the Decart production

$$\prod_{\alpha\in\Gamma^c} (T_\alpha^{(0)} \times T_\alpha^{(0)}) \bigcup (T_\alpha^{(1)} \times T_\alpha^{(1)}) \prod_{\alpha\in\Gamma} ([0,T_\alpha]\times[0,T_\alpha])$$

and equal to zero outside of this set. Notice, that $B_\varnothing(t,\tau) = B(t,\tau)$.

Let us suppose that

$$B(t,\tau) = B_{1\overset{(m)}{\dots}1}(t,\tau) = \prod_{j=1}^{m} B^{(j)}(t_j,\tau_j),$$

and the correlation function of the interpolating measure μ_0 has the form

$$B_0(t,\tau) = B_{0\overset{(m)}{\dots}0}(t,\tau) = \prod_{j=1}^{m} B_0^{(j)}(t_j,\tau_j)$$

At the fist step we applying the formula of measure interpolating N_1 times with respect to first argument of the function B and get

$$\int\limits_X F(x)d\mu(x) \approx \sum_{k_1=0}^{N_1} \frac{1}{2^{k_1}k_1!} \int\limits_0^T \overset{(2mk_1)}{\cdots} \int\limits_0^T \prod_{j=1}^{k_1} (B - B_{01\overset{(m-1)}{\dots}1})(t_j^{(1)},\tau_j^{(1)}) \times$$
$$\times \int\limits_X F^{(2k_1)}(x; t_1^{(1)},\tau_1^{(1)},\dots,t_{k_1}^{(1)},\tau_{k_1}^{(1)}) d\mu_{01\overset{(m-1)}{\dots}1}(x) d^m t_1^{(1)} d^m\tau_1^{(1)} \dots d^m t_{k_1}^{(1)} d^m \tau_{k_1}^{(1)}$$

At the next step for integrals of functionals of the form $F^{(2k_1)}(x;t_1^{(1)},\tau_1^{(1)},\ldots,t_{k_1}^{(1)},\tau_{k_1}^{(1)})$ we make the transition from measure $\mu_{01\overset{(m-1)}{\ldots}1}$ to $\mu_{001\overset{(m-2)}{\ldots}1}$ and get

$$\int\limits_X F^{(2k_1)}(x;t_1^{(1)},\tau_1^{(1)},\ldots,t_{k_1}^{(1)},\tau_{k_1}^{(1)})d\mu_{01\overset{(m-1)}{\ldots}1}(x)\approx$$

$$\approx\sum_{k_2=0}^{N_2}\frac{1}{2^{k_2}k_2!}\int\limits_0^T\overset{(2mk_2)}{\cdots}\int\limits_0^T\prod_{j=1}^{k_2}(B_{01\overset{(m-1)}{\ldots}1}-B_{001\overset{(m-2)}{\ldots}1})(t_j^{(2)},\tau_j^{(2)})\times$$

$$\times\int\limits_X F^{(2(k_1+k_2))}(x;t_1^{(1)},\tau_1^{(1)},\ldots,t_{k_1}^{(1)},\tau_{k_1}^{(1)},t_1^{(2)},\tau_1^{(2)},\ldots,t_{k_2}^{(2)},\tau_{k_2}^{(2)})d\mu_{001\overset{(m-2)}{\ldots}1}(x)\times$$

$$\times d^m t_1^{(2)}d^m\tau_1^{(2)}\ldots d^m t_{k_2}^{(2)}d^m\tau_{k_2}^{(2)}$$

In result, after repetitions for each of m variables we have

$$\int\limits_X F(x)d\mu(x)\approx$$

$$\approx\sum_{k_1,\ldots,k_m=0}^{N_1,\ldots,N_m}\frac{1}{2^{k_1+\ldots+k_m}k_1!\ldots k_m!}\int\limits_0^T\overset{(2m(k_1+\ldots+k_m))}{\cdots}\int\limits_0^T\prod_{j_1=1}^{k_1}(B-B_{01\overset{(m-1)}{\ldots}1})(t_{j_1}^{(1)},\tau_{j_1}^{(1)})\times$$

$$\times\prod_{j_2=1}^{k_2}(B_{01\overset{(m-1)}{\ldots}1}-B_{001\overset{(m-2)}{\ldots}1})(t_{j_2}^{(2)},\tau_{j_2}^{(2)})\ldots\prod_{j_m=1}^{k_m}(B_{0\overset{(m-1)}{\ldots}01}-B_{0\overset{(m)}{\ldots}0})(t_{j_m}^{(m)},\tau_{j_m}^{(m)})\times$$

$$\times\int\limits_X F^{(2(k_1+\ldots+k_m))}(x;t_1^{(1)},\tau_1^{(1)},\ldots,t_{k_1}^{(1)},\tau_{k_1}^{(1)},\ldots,t_1^{(m)},\tau_1^{(m)},\ldots,t_{k_m}^{(m)},\tau_{k_m}^{(m)})d\mu_0(x)\times$$

$$\times d^m t_1^{(1)}d^m\tau_1^{(1)}\ldots d^m t_{k_m}^{(m)}d^m\tau_{k_m}^{(m)}$$

Let us denote $(B-B_0)^{(j)}(t_{1j}^{(1)},\tau_{1j}^{(1)})=B^{(j)}(t_{1j}^{(1)},\tau_{1j}^{(1)})-B_0^{(j)}(t_{1j}^{(1)},\tau_{1j}^{(1)})$.

Using the representation of $B_{0\overset{(l)}{\ldots}01\overset{(m-l)}{\ldots}1}(t,\tau)$ we can write this formula in the next form

$$\int\limits_X F(x)d\mu(x)\approx\sum_{k_1,\ldots,k_m=0}^{N_1,\ldots,N_m}\frac{1}{2^{k_1+\ldots+k_m}k_1!\ldots k_m!}\times$$

$$\times\int\limits_0^T\overset{(2m(k_1+\ldots+k_m))}{\cdots}\int\limits_0^T\prod_{r=1}^{m}\prod_{j=1}^{k_r}\left((B-B_0)^{(r)}(t_{jr}^{(r)},\tau_{jr}^{(r)})\prod_{i=1}^{r-1}B_0^{(i)}(t_{ji}^{(r)},\tau_{ji}^{(r)})\prod_{l=r+1}^{m}B^{(l)}(t_{jl}^{(r)},\tau_{ji}^{(r)})\right)\times$$

$$\times\int\limits_X F^{(2(k_1+\ldots+k_m))}(x;t_1^{(1)},\tau_1^{(1)},\ldots,t_{k_1}^{(1)},\tau_{k_1}^{(1)},\ldots,t_1^{(m)},\tau_1^{(m)},\ldots,t_{k_m}^{(m)},\tau_{k_m}^{(m)})d\mu_0(x)\times$$

$$\times d^m t_1^{(1)}d^m\tau_1^{(1)}\ldots d^m t_{k_m}^{(m)}d^m\tau_{k_m}^{(m)},$$

here we suppose, that the production on i equal to 1 for $r=1$, the production on l equal to 1 for $r=m$ and the production on j equal to 1 for $k_r=0$.

Example.

Table 1: Exact values of integrals and corresponding approximated values

T	0.1	0.5	1	1.5	2	3
Exact	0.999988	0.994342	0.934571	0.769928	0.524929	0.122357
3D	0.985675	0.895407	0.840994	0.874931	0.871274	0.872246
Interp. + 3D	0.999412	0.980833	0.895494	0.723634	0.503802	0.153293

Let us consider the measure on the space $X[0,T] = C_2[0,T]$ (here $m=2$) with the correlation function $B^{(j)}(t_j,\tau_j) = \exp\{-\alpha_j|t_j-\tau_j|\}$, $j=1,2$. We need to calculate integral for the functional $F(x) = \exp\{\lambda \boldsymbol{i} \int\limits_0^T \int\limits_0^T x(t,\tau)dtd\tau\}$.

The functional derivative of the n degree for this functional is

$$F^{(n)}(x;t_1,\tau_1,\dots,t_n,\tau_n) = (\lambda \boldsymbol{i})^n F(x).$$

The excat value of considered integral equal to

$$I(T) = \int\limits_{X[0,T]} F(x)d\mu(x) = \exp\left\{ -\frac{1}{2}\lambda^2 \prod_{j=1}^{2} \frac{2}{\alpha_j^2}(\mathrm{e}^{-\alpha_j T} - 1 + \alpha_j T)\right\}.$$

For calculation of the approximate value we use the formula of the third degree of accuracy (see [4]):

$$J(T) = \alpha_1\alpha_2 \int\limits_{\mathbf{R}^2} \mathrm{e}^{-2(\alpha_1|u_1|+\alpha_2|u_2|)} F(\rho(u,\cdot))d^2u,$$

where $\rho(u,s) = \prod\limits_{j=1}^{2} \rho_j(u_j,s_j)$, $\quad \rho_j(u_j,s_j) = \begin{cases} \mathrm{e}^{\alpha_j s_j}\mathrm{sign}(u_j), & s_j < |u_j|, \\ 0, & s_j \geqslant |u_j|. \end{cases}$

The result of calculation is given in the TABLE 1.

3. Formula without derivatives of integrated functional

Now let us suppose $X = X([0,T]\times[0,T])$ — Gilbert space of functions on $[0,T]\times[0,T]$ and μ — Gaussian measure with mean value equal to zero and with correlation function $B(t_1,t_2;\tau_1,\tau_2) = R(t_1,\tau_1)R(t_2,\tau_2)$, $R(t,\tau) = p(\min(t,\tau))q(\max(t,\tau))$, where

$$\int\limits_0^T \int\limits_0^T R(\tau_1,\tau_2)a_k(\tau_1)a_j(\tau_2)d\tau_1 d\tau_2 = \lambda_k\delta_{k,j}, \;\; k,j = \overline{1,m}.$$

In this section we suppose that

$$F(x) = G\left(\int\limits_0^T x(\cdot,\tau)a_1(\tau)d\tau, \dots, \int\limits_0^T x(\cdot,\tau)a_m(\tau)d\tau\right),$$

where F — continuous on R^m, bounded by some functional, which is integrable with respect to measure μ.

As in previous section we calculate the integral

$$I \equiv \int_X F(x)d\mu(x).$$

Here we suppose that $\{\varphi_i(t)\}_{i=1}^{\infty}$ — orthonormal eigen functions of kernel $R(t,s)$ on $[0,T/2]$, $\{\psi_i(t)\}_{i=1}^{\infty}$ — orthonormal eigen functions of kernel $R(t,s)$ on $[T/2,T]$.

Let us denote $X_1 = X([0,T/2]\times[0,T])$, $X_2 = X([T/2,T]\times[0,T])$. For constructing approximation formula we use the representation $x(t,\tau) = \widehat{x}(t,\tau) + \widetilde{x}(t,\tau)$, where

$$\widehat{x}(t,\tau) = \lim_{n\to\infty} \widehat{x}_n(t,\tau) \equiv \lim_{n\to\infty} \sum_{i=1}^{n} \int_0^{T/2} \widehat{x}(s,\tau)\varphi_i(s)ds\ \varphi_i(t)$$

$$\widetilde{x}(t,\tau) = \lim_{n\to\infty} \widetilde{x}_n(t,\tau) \equiv \lim_{n\to\infty} \sum_{i=1}^{n} \int_{T/2}^{T} \widetilde{x}(s,\tau)\psi_i(s)ds\ \psi_i(t).$$

Theorem. Interpolation formula has the form

$$I = \int_{X_1}\int_{X_2} F(\widehat{x}(\cdot)+\widetilde{x}(\cdot))d\mu(\widehat{x})d\mu(\widetilde{x}) + \sum_k \frac{1}{k!}\sum_{r_1}^{m}\dots\sum_{r_k}^{m}\int_{X_1}\int_{X_2} F(\widehat{x}(\cdot)+\widetilde{x}(\cdot))\times$$

$$\times H^{(k)}(\widehat{x}(\cdot);\delta_{T/2}(\cdot)/q(T/2),a_{r_1}(\cdot),\dots,\delta_{T/2}(\cdot)/q(T/2),a_{r_k}(\cdot))\times$$

$$\times H^{(k)}(\widetilde{x}(\cdot);\delta_{T/2}(\cdot)/p(T/2),a_{r_1}(\cdot),\dots,\delta_{T/2}(\cdot)/p(T/2),a_{r_k}(\cdot))d\mu(\widehat{x})d\mu(\widetilde{x})$$

where $\delta_{T/2}(t) = \delta\left(t-\frac{T}{2}\right)$

$H^{(k)}$ — functional Hermite polynomials.

It is easy to proof this theorem by using Fourie transformation of the density of the measure μ and properties of Hermite polynomials.

4. Summarizing conclusions

Usually, direct applying of approximated formulas give us good result of calculations for the case, when T is small and very bad results when T is large. The used method allows us pass from integration on spaces of functions on T to integration on spaces of functions on smaller region and after such transformation we can use formulas of desired accuracy.

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