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Laurent series solution of the Skyrme model

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The Painlevé property of the hedgehog configuration of the Skyrme model is investigated with the aid of the the Ablowitz-Ramani-Segur algorithm (Painlevé test). The field equation seems to pass the Painlevé test leaving two free parameters in the solutions of Laurent-series-type. The convergence property of solutions of Laurent-series-type is examined numerically up to 500th order. The series seems to have a finite radius of convergence. The profile function of the model is investigated and static energy of the model is numerically calculated.

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1. Introduction

The Skyrme model [1] is known to be an effective theory of Quantumchromodynamics (QCD) [2, 3] and admits stable soliton solutions [4]. It is defined by the Lagrangian density

$$\mathcal{L}_S = \frac{F_\pi^2}{16}\mathrm{tr}(\partial_\mu U)(\partial^\mu U^\dagger) + \frac{1}{32e^2}\mathrm{tr}\left([\partial_\mu U U^\dagger\,,\,\partial_\nu U U^\dagger][\partial^\mu U U^\dagger\,,\,\partial^\nu U U^\dagger]\right), \tag{1}$$

where $U = U(x)$ is an element of $SU(2)$ and F_π and e are constants should be fixed by comparison with experimental data. If we introduce $su(2)$-valued current $R_\mu = (\partial_\mu U)U^\dagger$, the field equation is written by

$$\partial_\mu\left(R^\mu + \frac{1}{4}\left[R^\nu,\ \left[R_\nu,\ R^\mu\right]\right]\right) = 0 \tag{2}$$

and static energy is defined as

$$E = \frac{1}{12\pi}\int\left[-\frac{1}{2}\mathrm{tr}\left(R_i R_i\right) - \frac{1}{16}\mathrm{tr}\left(\left[R_i,\ R_j\right]\left[R_i,\ R_j\right]\right)\right] d^3x, \tag{3}$$

where we rescaled length and energy as $F_\pi/4e$ and $2/eF_\pi$. To discuss soliton solution of the model, an important topological number, baryon number B is defined by

$$B = \frac{\varepsilon_{ijk}}{24\pi^2}\int \mathrm{tr}\left(R_i R_j R_k\right) d^3x. \tag{4}$$

The soliton solutions of the Skyrme model are classified by the baryon number B. Battye and Suttcliffe [4] illuminated that numerical analysis of this model reveals that it admits polyhedral soliton solutions for each baryon number.

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If we impose the spherically symmetric(hedgehog) *Ansatz*

$$U = \exp\left[if(r)\left(\hat{\boldsymbol{x}}\cdot\boldsymbol{\tau}\right)\right], \quad r = |\boldsymbol{x}|, \tag{5}$$

the field equation becomes the following ordinary differential equation

$$\left[r^2 + 2\sin^2 f(r)\right]\frac{d^2 f(r)}{dr^2} + 2r\left(\frac{df(r)}{dr}\right) + \sin 2f(r)\left[\left(\frac{df(r)}{dr}\right)^2 - 1 - \frac{\sin^4 f(r)}{r^2}\right] = 0, \tag{6}$$

where $r = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2}$. It seems difficult to obtain the analytic solutions for the equation. Cho[5] noted that the constant configuration $f(r) = 2\pi$ solves the Eq.(6) and constitutes the monopole solution of the Skyrme model.

Introducing the variables

$$z = \frac{r^2}{r^2+2}, \quad v(z) \equiv \tan^2 f(z), \tag{7}$$

field equation for the Skyrme model(6) can be rewritten as

$$\frac{d^2v}{dz^2} - \frac{1}{2}\left[\frac{3}{v+1} + \frac{1}{v} - \frac{1}{v+z}\right]\left(\frac{dv}{dz}\right)^2 + \frac{1}{2}\left[\frac{1}{z-1} + \frac{1}{z} + \frac{2}{v+z}\right]\frac{dv}{dz} + \frac{v\left[v(z+1)+2z\right]}{2z^2\left(z-1\right)^2\left(v+z\right)} = 0. \tag{8}$$

For this equation, we obtained the solution as a Laurent series of the form[7]

$$v(z) = \frac{z\left(1-z\right)^2}{\left(z-z_0\right)^2}\sum_{j=0}^{\infty} w_j\left(z-z_0\right)^j, \tag{9}$$

where z_0 is an arbitrary parameter. Another free parameter in this solutions is w_0. If we input values z_0 and w_0, all $w-j$s ($j \geq 1$) are expressed by z_0 and w_0 without ambiguity. In this paper, we will review the solution Eq.(9) with the help of ARS algorithm[6]. This paper is organized as follows. In Sec.2 we will discuss the Painlevé property of the Skyrme model applying ARS algorithm to the field equation (8). In Sec.3 the behavior of the profile function $f(r)$ and a least energy E by $f(r)$ is investigated. We shall close by giving a brief summary in Sec.4.

2. Painlevé test for the Skyrme model

Ablowitz, Ramani and Segur (ARS)[6] proposed a criterion to judge whether any solution of a given ordinary differential equation has movable branch points or not. To analyze the Painlevé property of Eq(8), it is convenient to rewrite Eq.(8) as

$$\begin{aligned} &2v\left(v-1\right)\left(z-v\right)\frac{d^2v}{dz^2} - \left(3v^2 - 4zv + z\right)\left(\frac{dv}{dz}\right)^2 + \frac{v\left(v-1\right)}{z\left(z-1\right)}\left(4z^2 - 2zv - 3z + v\right)\frac{dv}{dz} \\ &+ \frac{v^2\left(v-1\right)}{z^2\left(z-1\right)^2}\left[v\left(z+1\right) - 2z\right] = 0. \end{aligned} \tag{10}$$

We suppose that

$$v(z) = \sum_{j=0}^{\infty} v_j\left(z-z_0\right)^{j-\alpha} \tag{11}$$

is a solution of Eq.(8) around $z = z_0$ and assuming that z_0 is different from 0 and 1.

2.1. Leading order analysis

Substituting $v(z) = v_0 (z - z_0)^{-\alpha}$ into the leading terms in Eq.(10),

$$-2v^3 \frac{d^2 v}{dz^2} + 3v^2 \left(\frac{dv}{dz} \right)^2, \tag{12}$$

and balancing as

$$2\alpha (\alpha - 1) v_0^4 P^{-(4\alpha+2)} - 3\alpha^2 v_0^4 P^{-(4\alpha+2)} = 0,$$

we obtain the *leadingorder* $\alpha = 2$ and v_0 is arbitrary where $P \equiv (z - z_0)$.

2.2. Resonance analysis

To find resonances it is enough to analyze the leading order terms. Substituting the Laurent series

$$v(z) = \sum_{j=0}^{\infty} v_j P^{j-2} \quad (v_0 \neq 0) \tag{13}$$

into Eq.(10), we obtain the relation

$$\sum_{k=0}^{\infty} P^k [k(k+1) v_k + F_k(v_0, \cdots, v_{k-1})] = 0, \tag{14}$$

where $F_k(v_0, ..., v_{k-1})$ is a function of $v_0, \cdots, v_{k-1}$. From the coefficient of v_k, we obtain the *resonances* $k = -1$ and 0. Here, $k = -1$ and $k = 0$ correspond to the arbitrariness of z_0 and v_0, respectively.

2.3. Compatibility check

The coefficients $v_1, v_2, \cdots$ are calculated as

$$\begin{aligned}
v_0 &= \text{arbitrary}, \quad v_1 = \frac{-v_0 + 2v_0 z_0}{2z_0(z_0 - 1)}, \\
v_2 &= \frac{v_0 + 4v_0 z_0^2 - 16z_0^2(z_0 - 3)(z_0 - 1)^2}{48z_0^2(z_0 - 1)^2}, \\
v_3 &= \frac{-32z_0^3(z_0 - 1)^3 + v_0(1 - 2z_0 - 4z_0^2)}{96z_0^3(z_0 - 1)^3}, \cdots .
\end{aligned}$$

We computed v_j's up to v_{500} numerically in some cases of (v_0, z_0). The radius of convergence R of the Laurent series is determined by $R = \lim_{j\to\infty} \left(|v_j|^{1/j} \right)^{-1}$. The result of the $v_0 = -1, z_0 = 1/2$ case is given in Fig. 1.

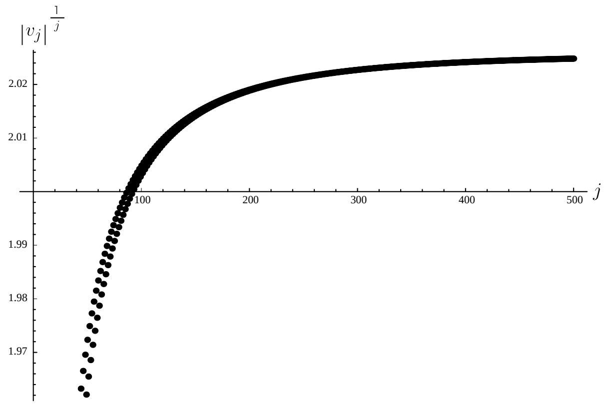

Fig.1: $|v_j|^{1/j}$ in the case $v_0 = -1$ and $z_0 = 1/2$.

This result is summarized as $|v_{500}|^{1/500} \sim 2.0248\cdots$. The radius of convergence could be surmised as $\approx 1/2$ since the value of $|v_j|^{1/j}$ hardly changes for $j \lesssim 500$ as is shown in Table 1.

Table 1: $|v_j|^{1/J}$ for j=490 to 500

j	490	491	492	493	494	495	496	497	498	499	500
$\|v_j\|^{1/j}$	2.02478	2.02478	2.02478	2.02479	2.02479	2.0248	2.0248	2.02481	2.02481	2.02481	2.02482

2.4. Behavior at spatial origin and infinity

Since we assumed that $z_0 \neq 0, 1$ in analysis of the solution (10), we here consider the case $z_0 = 0$ in (11):

$$v(z) = \sum_{j=0}^{\infty} v_j z^{j-\alpha}, \tag{15}$$

which corresponds spatial origin. In this case, leading behavior of the field equation (10) depends on values of α. For example, (10) reduces as

$$-2v_0^4 z^{4\alpha}\alpha\left(\alpha - 1\right) + 3v_0^4 z_{4\alpha}\alpha^2 - \alpha v_0^4 z^{4\alpha} + v_0^4 z^{4\alpha} = 0 \tag{16}$$

for $\alpha > 0$ case and there are no real α satisfying $\alpha > 0$. Through the leading order analysis for $\alpha < 0, \alpha = 0, 0 < \alpha < 1, \alpha = 1, \alpha > 1$ cases, we conclude that $\alpha = -1$ is consistent.

Next, we consider the case $z_0 = 1$ in (11):

$$v(z) = \sum_{j=0}^{\infty} v_j \left(z - 1\right)^{j-\alpha}, \tag{17}$$

which corresponds spatial infinity. Leading order analysis again for this case results $\alpha = -2$.

Since the solution (11) for each cases are the form of infinite series, $v(z)$ of the form (9) is also a solution for the field equation (10). Note that the behavior of $v(z)$ matches the boundary condition

$$f(0) = \pi, \quad f(\infty) = 0. \tag{18}$$

Substituting the solution (9) into Eq.(10), The coefficients $w_1, w_2, \cdots$ are determined except w_0 as

$$\begin{aligned}
w_0 &= \text{arbitrary}, \quad w_1 = \frac{v_0(4z_0-1)}{2z_0(1-z_0)}, \quad w_2 = \frac{(148z_0^2-72z_0+25)w_0-16z_0(z_0-3)}{48z_0^2(1-z_0)^2}, \\
w_3 &= \frac{(51-200z_0+292z_0^2-408z_0^3)w_0-32z_0(2z_0^2-9z_0+3)}{96z_0^3(z_0-1)^3}, \\
w_4 &= \frac{1}{3840w_0z_0^4(z_0^4-1)^4}\left[(21104z_0^4-19840z_0^3+20176z_0^2-10200z_0+2093)w_0^2\right. \\
&\quad \left. -(3776z_0^4-23008z_0^3+15260z_0^2+3936z_0)w_0+768z_0^3-512z_0^4\right]. \qquad (19)
\end{aligned}$$

We computed w_j's up to w_{500} numerically as same in Sec C. The result of the $v_0=-2, z_0=1/2$ case is given in Fig. 2.

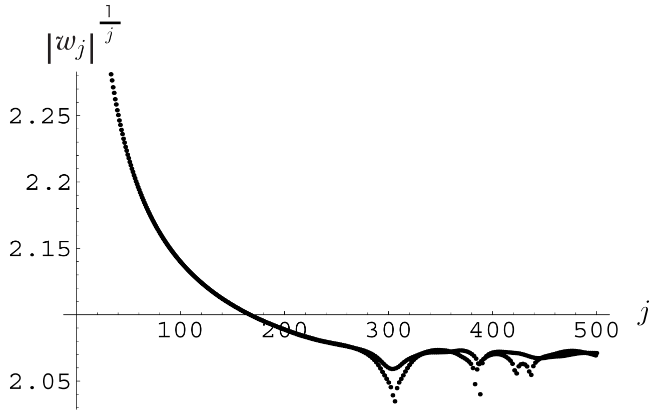

Fig.2: $|w_j|^{1/j}$ in the case $w_0=-2$ and $z_0=1/2$.

This result is summarized as $|v_{500}|^{1/500} \sim 2.07112\cdots$. The radius of convergence could be surmised as $\approx 1/2$ and values of $j \lesssim 500$ as is shown in Table 1.

Table 2: $|w_j|^{1/J}$ for j=490 to 500

j	490	491	492	493	494	495	496	497	498	499	500
$|w_j|^{1/j}$	2.07126	2.07106	2.07124	2.0707	2.07121	2.07029	2.07119	2.06982	2.07116	2.06931	2.07112

3. Profile function $f(r)$ and Static energy

From the discussion in previous subsection, $v(z)$ takes the form

$$v(z) = \frac{z(1-z)^2}{(z-z_0)^2}w(z), \qquad (20)$$

where $w(z)$ is a smooth function which is non-vanishing and finite for $0 \leqq z \leqq 1$. Then we see that the natural relation between $f(r)$ and $v(z)$ inverse to Eq.(7) is given by

$$f(r) = \begin{cases} \pi - \mathrm{Arctan}\sqrt{v(z)}, & 0 \leqq z \leqq z_0, \\ \mathrm{Arctan}\sqrt{v(z)}, & z_0 \leqq z \leqq 1. \end{cases} \qquad (21)$$

We have numerically estimated E with changing w_0 and z_0 by the step 0.001 with setting $w(z)$ as

$$w(z) = \sum_{k=0}^{N} w_k (z - z_0)^k, \quad N\text{:small}, \tag{22}$$

and find that

$$E = 1.23186 \cdots \cong 1.2319 \tag{23}$$

for

$$v(z) = \frac{0.673z\,(1-z)^2}{(z-0.279)^2} \tag{24}$$

is at least a local minimum[7]. The profile function and energy density of this case is depicted in Fig.1 and Fig.2, respectively.

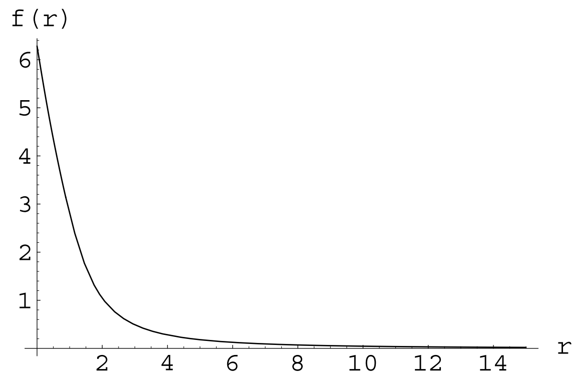

Fig.1: Profile function $f(r)$ for $w_0 = 0.673, z_0 = 0.279$.

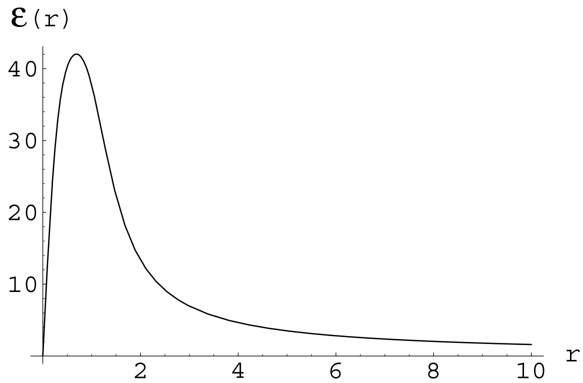

Fig.2: Energy density $\varepsilon(r)$ for $w_0 = 0.673, z_0 = 0.279$.

In the $N = 1, 2, 3$ cases, with the help of (19), we have

$$\begin{cases} N=1, \quad w_0 = 0.670, \quad z_0 = 0.275, \quad E = 1.23215\cdots \cong 1.2322. \\ N=2, \quad w_0 = 0.485, \quad z_0 = 0.327, \quad E = 1.34000\cdots \cong 1.3400. \\ N=3, \quad w_0 = 0.385, \quad z_0 = 0.249, \quad E = 1.34234\cdots \cong 1.3423. \end{cases} \tag{25}$$

We see that the $N = 0$ case gives a smaller E than $N = 1, 2, 3$ cases.

4. Conclusion

We have explored an analytical solution for the Skyrme model under spherically symmetric *Ansatz* (8). Our solution is of Laurent-series-type which contains two arbitrary parameters z_0 and w_0. The arbitrariness of z_0 and w_0 corresponds to the resonances $j = -1$ and 0, respectively. We have not encountered any inconsistency of the expansion of $w(z)$ in a Laurent series. We checked that our solution has finite radius of convergence. We have numerically obtained $R \sim 1/2$ for $z_0 = 1/2$ and $w_0 = -2$. This R indicates that our solution $f(z)$ is regular in $0 < r < \infty$ since $0 < r < \infty$ corresponds to $0 < z < 1$ by (7). We have numerically estimated E with changing w_0 and z_0 and find that the smallest energy can be calculated in $N = 0$ case.

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