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# The Majorana bases structure and an electrically neutral spin 1/2 particle in Riemannian space-time

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The case of electrically neutral spin 1/2 particle is investigated on the background of a curved space-time. In any one of Majorana bases the Dirac generally covariant equation reduces to a pair of separate equations for a real and complex parts of the 4-spinor wave function $\Psi = \varphi_+ + i\varphi_-$. The lorentz matrices in bispinor space, being specified in any Majorana basis, turn to be real-valued, therefore equations for $\varphi_+$ and $i\varphi_-$ are not mixed by local Lorentz transformations associated with tetrad changes in curved space. The set of all Majorana bases is looked through in detail, and 17-parametric transformation $\Psi_M(x) = A(m_i, n_i, e^{i\alpha})\ \Psi(x)$ referring spinor basis to any Majorana's is derived. Several the simplest examples of choosing parameters $(m_i, n_i, e^{i\alpha})$ are given, they correspond to widely used Majorana representations. One special case of electrically neutral particle with non-vanishing magnetic moment is discussed. Its wave function must be complex-valued. In other words, real nature of the wave function of the particle is equivalent to requirement – all electromagnetic characteristics of the particle, including electric charge, must vanish.

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## 1. Majorana fermion in a curved space-time

Let us consider a generally covariant Dirac equation [1,2]

$$\{ \ \gamma^{\alpha}(x) \ [ \ i \ ( \ \partial_{\alpha} \ + \ \Gamma_{\alpha}(x) \ ) \ - eA_{\alpha} \ ] - \ m \ \} \ \Psi(x) = 0 \ , \tag{1}$$
$$\gamma^{\alpha}(x) = \gamma^{a} \ e^{\alpha}_{(a)}(x) \ , \qquad \Gamma_{\alpha}(x) = \frac{1}{2} \ \sigma^{ab} \ e^{\beta}_{(a)} \ \nabla_{\alpha} \ ( \ e^{\alpha}_{(b);\beta} \ )$$

in one of Majorana bases [3] of bispinor space which is characterized by the properties [4]:

$$(i \ \gamma^{a}_{M})^{*} = +\gamma^{a}_{M} \ , \qquad (\sigma^{ab}_{M})^{*} = +\sigma^{ab}_{M} \ , \qquad (i \ \gamma^{5}_{M})^{*} = +\gamma^{5}_{M} \ . \tag{2}$$

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The wave Dirac operator (1) in any curved space in absence of electromagnetic field, being specified for any Majorana basis turns to be real-valued

$$[\, i\gamma^{\alpha}(x)\,(\,\partial_{\alpha} \,+\, \Gamma_{\alpha}(x)\,) \,-\, m\,]^{*} = \, i\gamma^{\alpha}(x)\,(\,\partial_{\alpha} \,+\, \Gamma_{\alpha}(x)\,) \,-\, m \;\; .$$

This means that real and imaginary parts of the wave function $\Psi_M = \varphi_+ + i\,\varphi_-$ are not mixed by the gravitational terms. In other words, there are two separate equations for the fields $\varphi_+$ and $\varphi_-$:

$$\begin{gathered}[\, i\gamma^{\alpha}(x)\,(\,\partial_{\alpha} \,+\, \Gamma_{\alpha}(x)\,) \,-\, m\,]\varphi_+ = 0\;, \\ [\, i\gamma^{\alpha}(x)\,(\,\partial_{\alpha} \,+\, \Gamma_{\alpha}(x)\,) \,-\, m\,]\varphi_- = 0\;. \end{gathered} \tag{3}$$

If one starts with another form of the Dirac equation [2]

$$\begin{gathered} B_k(x) = \frac{1}{2} e^{\alpha}_{(k)\,;\alpha}(x)\;, \qquad C_k(x) = \frac{1}{4}\epsilon^{abc}{}_{k}\gamma_{abc}(x)\;, \\ \{\gamma^{k}\,[\,i\,(\,e^{\alpha}_{(k)}\partial_{\alpha} \,+\, B_k \,-\, i\gamma^{5}C_k\,) \,-\, eA_a\,] \,-\, m\}\;\Psi = 0\;. \end{gathered} \tag{4}$$

the same result will be obtained: equations for the real and imaginary parts of the $\Psi$ are independent from each other:

$$\begin{gathered} \Big[\, i\,\gamma^{k}\,(\,e^{\alpha}_{(k)}\partial_{\alpha} \,+\, B_k \,-\, i\gamma^{5}C_k\,) \,-\, m\,\Big]\;\varphi_+ = 0\;, \\ \Big[\, i\gamma^{k}\,(\,e^{\alpha}_{(k)}\partial_{\alpha} \,+\, B_k \,-\, i\gamma^{5}C_k\,) \,-\, m\,\Big]\;\varphi_- = 0\;. \end{gathered} \tag{5}$$

Thus, two Majorana fermion fields, real $\varphi_+(x)$ and imaginary $i\,\varphi_-$, can interact with gravitational field according (3) (or (5)).

## 2. On Lorentz 4-spinor transformations in Majorana basis

The Lorentz transforms for 4-spinor wave functions have the simplest forme in 2-spinor representation [4]

$$\Psi(x) = (\xi(x),\, \eta(x)), \qquad \xi' = B(k)\xi, \qquad \eta' = B(\bar{k}^{*})\eta\;. \tag{6}$$

To obtain explicit expressions for 4-spinor Lorentz matrix in arbitrary basis it suffices to decompose the above matrix $S = B(k) \oplus B(\bar{k}^{*})$ in terms of sixteen elementary matrices: $I,\ \gamma^5,\ \gamma^a,\ \gamma^5\gamma^a,\ \sigma^{ab}$. The coefficients of that decomposition do not depend on (accidental) choice of the Dirac matrices. In that way, let us represent $S(k,\ k^*)$ in spinor basis

$$S(k,\ k^{*}) = \left| \begin{array}{cc} \sigma^{a}\,k_a & 0 \\ 0 & \bar{\sigma}^{a}\,k_a \end{array} \right| \tag{7}$$

as a linear combination

$$S = \Phi\, I \,+\, \tilde{\Phi}\,\gamma^{5} \,+\, \Phi_a\,\gamma^{a} \,+\, \tilde{\Phi}_a\,\gamma^{5}\,\gamma^{a} \,+\, \Phi_{ab}\,\sigma^{ab}\;, \tag{8}$$

and further

$$\begin{vmatrix} \sigma^a\, k_a & 0 \\ 0 & \bar{\sigma}^a\, k_a \end{vmatrix} = \Bigg[ \Phi \begin{vmatrix} I & 0 \\ 0 & I \end{vmatrix} + \tilde{\Phi} \begin{vmatrix} -I & 0 \\ 0 & +I \end{vmatrix} +$$

$$\Phi_a \begin{vmatrix} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{vmatrix} + \tilde{\Phi}^a \begin{vmatrix} 0 & -\bar{\sigma}^a \\ \sigma^a & 0 \end{vmatrix} + \Phi^{ab} \begin{vmatrix} \Sigma^{ab} & 0 \\ 0 & \bar{\Sigma}^{ab} \end{vmatrix} \Bigg] .$$

From the later one gets

$$0 = \Phi_a\, \bar{\sigma}^a - \tilde{\Phi}_a\, \bar{\sigma}^a , \qquad 0 = \Phi_a\, \sigma^a + \tilde{\Phi}_a\, \sigma^a ,$$
$$\sigma^a\, k_a = \phi - \tilde{\Phi} + \Phi_{ab}\, \Sigma^{ab} , \qquad \sigma^a\, k_a = \phi + \tilde{\Phi} + \Phi_{ab}\, \bar{\Sigma}^{ab} .$$

Evidently, $\Phi_a = 0$ and $\tilde{\Phi}_a = 0$; the remaining equations are readily solved with the use of trace technique for Dirac matrices:

$$k^c = (\Phi - \tilde{\Phi})\, g^{0c} + \Phi^{0c} - i/2\, \Phi_{ab}\, \epsilon^{abc0} ,$$
$$k^{*c} = (\Phi + \tilde{\Phi})\, g^{0c} + \Phi^{0c} + i/2\, \Phi_{ab}\, \epsilon^{abc0} . \qquad (9)$$

From (9) it follows:

$$\Phi = (k_0^* + k_0)/2 , \qquad \tilde{\Phi} = (k_0^* - k_0)/2 ,$$
$$\Phi_{01} = (k_1^* + k_1)/2 , \qquad \Phi_{23} = (k_1^* - k_1)/2i ,$$
$$\Phi_{02} = (k_2^* + k_2)/2 , \qquad \Phi_{31} = (k_2^* - k_2)/2i ,$$
$$\Phi_{03} = (k_3^* + k_3)/2 , \qquad \Phi_{12} = (k_3^* - k_3)/2i . \qquad (10)$$

Turning to eq. (8), one arrives at

$$S(k,\ k^*) = \frac{1}{2}(k_0 + k_0^*) - \frac{1}{2}(k_0 - k_0^*)\gamma^5 + k_1(\sigma^{01} + i\sigma^{23}) + k_1^*(\sigma^{01} - i\sigma^{23}) +$$
$$k_2(\sigma^{02} + i\sigma^{31}) + k_2^*(\sigma^{02} - i\sigma^{31}) + k_3(\sigma^{03} + i\sigma^{12}) + k_3^*(\sigma^{03} - i\sigma^{12}) . \qquad (11)$$

Introducing real and imaginary parts in complex $k_a = m_a - in_a$, the previous relation can be written as

$$S(m_a, n_a) = (m_0 + n_0 i\gamma^5) +$$
$$+(m_1\sigma^{01} + m_2\sigma^{02} + m_3\sigma^{03}) + (n_1\sigma^{23} + n_2\sigma^{31} + n_3\sigma^{12}) \qquad (12)$$

The formula obtained (12) provides us with explicit form of Lorentz transformations for 4-spinor wave function, it is the same in all bases. Specifying the matrices involved according to any of Majorana forme $(i\gamma^5_M)^* = i\gamma^5_M$ , $(\sigma^{ab}_M)^* = +\,\sigma^{ab}_M$ , we see that the Lorentz transformations are real-valued in those bases.

This property is of primary significance in the context of symmetry properties of the Majorana equation under Lorentz group in any curved space-time. In is known that in any Riemannian space-time the Dirac equation for a charged fermion proves gauge symmetry under local Lorentz group, which is related with the freedom to choose an arbitrary tetrad $e^{\beta}_{(a)}(x)$ at a given space-time metric $g_{\alpha\beta}(x)$. The property of bispinor Lorentz matrices to be real-valued in Majorana representations means that the local Lorentz gauge symmetry for Majorana equation holds as well.

## 3. On Majorana bases structure

Now we are to describe all possible Majorana's bases. To this end we should find all transformations $A$ in 4-spinor space that change all the Dirac matrices to the imaginary forme

$$\Psi_M(x) = A\ \Psi(x)\ , \qquad \gamma^a_M = A\ \gamma^a\ A^{-1}\ , \qquad (\gamma^a_M)^* = -\gamma^a_M\ ; \tag{13}$$

here $\gamma^a$ stand for the Dirac matrices in spinor representation. One should note that the problem must have a many of solutions. Indeed, if $A$ satisfies eq. (13) then any matrix of the form $A' = e^{i\alpha}\ R\ A$, where $R$ is real, will satisfy eq. (13) as well:

$$\text{if} \qquad (A\ \gamma^a\ A^{-1})^* = -(A\ \gamma^a\ A^{-1})\ ,$$
$$\text{then} \qquad \Big[\ (e^{i\alpha}\ R\ A)\ \gamma^a\ (e^{i\alpha}\ R\ A)^{-1}\ \Big]^* = -\Big[\ (e^{i\alpha}\ R\ A)\ \gamma^a\ (e^{i\alpha}\ R\ A)^{-1}\ \Big]^*\ .$$

Equation (13) can be written as

$$A^*\ (\gamma^a)^*\ (A^*)^{-1} = -A\ \gamma^a\ A^{-1}\ , \qquad \text{or} \qquad (A^{-1}\ A^*)\ (\gamma^a)^*\ (A^{-1}\ A^*)^{-1} = -\ \gamma^a\ .$$

With the use of notation $A^{-1}\ A^* = U$, the latter reads

$$U\ (\gamma^a)^*\ U^{-1} = -\ \gamma^a\ . \tag{14}$$

In spinor representation four identities hold

$$(\gamma^0)^* = +\gamma^0\ , \ \ (\gamma^1)^* = +\gamma^1\ , \ \ (\gamma^2)^* = -\gamma^2\ , \ \ (\gamma^3)^* = +\gamma^3\ ,$$

therefore, solution of eq. (14) looks as

$$U = const\ \gamma^2\ , \qquad \det\ (\gamma^2) = +1\ . \tag{15}$$

Because $\det\ U = (det\ A)^*/(det\ A)$ , one must conclude that *const* is equal to a phase factor $e^{i\alpha}$. Thus, the problem is reduced to

$$A^{-1}\ A^* = e^{i\alpha}\ \gamma^2\ , \quad \text{or} \quad A^* = e^{i\alpha}\ A\ \gamma^2\ . \tag{16}$$

Any 4-dimensional matrix can be decomposed into sixteen Dirac elementary matrices:

$$\begin{array}{llllll} I & \gamma^0 & \gamma^1 & \gamma^2 & \gamma^3 & \\ \gamma^5 & \gamma^5\gamma^0 & \gamma^5\gamma^1 & \gamma^5\gamma^2 & \gamma^5\gamma^3 & \\ \gamma^0\gamma^1 & \gamma^0\gamma^2 & \gamma^0\gamma^3 & \gamma^1\gamma^2 & \gamma^1\gamma^3 & \gamma^2\gamma^3 \end{array} .$$

Let us take the $A$ in the form (this is a combination of the above sixteen ones)

$$\begin{aligned} A = & \Big[\ (M_0\gamma^2 + m_0)\ +\ \gamma^1\ (N_0\gamma^2 + n_0)\ \Big]\ + \\ & +\ \gamma^5\ \Big[\ (M_1\gamma^2 + m_1)\ +\ \gamma^1\ (N_1\gamma^2 + n_1)\ \Big]\ + \\ & +\ \gamma^0\ \Big[\ (M_2\gamma^2 + m_2)\ +\ \gamma^1\ (N_2\gamma^2 + n_2)\ \Big]\ + \\ & +\ \gamma^5\gamma^0\ \Big[\ (M_3\gamma^2 + m_3)\ +\ \gamma^1\ (N_3\gamma^2 + n_3)\ \Big]\ . \end{aligned} \tag{17}$$

With the notation

$$\Gamma_0 = I \; , \quad \Gamma_1 = \gamma^5 \; , \quad \Gamma_2 = \gamma^0 \; , \quad \Gamma_3 = \gamma^5 \gamma^0 \; , \tag{18}$$

eq. (17) is written in the abridged form

$$A = \Gamma_i \; \big[ \, (M_i \, \gamma^2 + m_i) \; + \; \gamma^1 \, (N_i \, \gamma^2 + n_i) \, \big] \; . \tag{19}$$

Now, let us substite (19) into eq. (16):

$$\begin{aligned} \Gamma_i \; \big[ \, (-M_i^* \gamma^2 + m_i^*) \; + \; \gamma^1 \, (-N_i^* \gamma^2 + n_i^*) \, \big] = \\ = e^{i\alpha} \Gamma_i \; \big[ \, (m_i \gamma^2 - M_i) \; + \; \gamma^1 \, (n_i \gamma^2 - N_i) \, \big] \; ; \end{aligned}$$

further it follow equations for unknown parameters:

$$M_i = -e^{-i\alpha} \, m_i^* \; , \qquad N_i = -e^{-i\alpha} \, n_i^* \; .$$

Therefore, expression for a matrix $A$, relating spinor basis with any one Majorana's is given by [1]

$$\begin{aligned} A = \Gamma_i \; \big[ \, (m_i - e^{-i\alpha} \, m_i^* \, \gamma^2) \; + \; \gamma_1 \, (n_i - e^{-i\alpha} \, n_i^* \, \gamma^2) \, \big] \; , \\ \Psi_M(x) = A(m_i, n_i, e^{i\alpha}) \; \Psi(x) \; . \end{aligned} \tag{20}$$

Evidently, 17 arbitrary real parameters enter these formulas; there exists one additional restriction, det $A \neq 0$.

## 4. Particular Majorana bases

Let us written down several simple bases of majorana: (only different from zero parameters are specified):

$$\begin{aligned} \underline{m_0 = 1/\sqrt{2}, \qquad e^{i\alpha} = +1}, \qquad A = \frac{1-\gamma^2}{\sqrt{2}} \; , \qquad A^{-1} = \frac{1+\gamma^2}{\sqrt{2}} \; , \\ \gamma^0_M = +\gamma^0 \gamma^2, \qquad \gamma^1_M = +\gamma^1 \gamma^2, \qquad \gamma^2_M = \gamma^2, \qquad \gamma^3_M = +\gamma^3 \gamma^2. \end{aligned} \tag{21}$$

There exist 16 such simple variants which can be represented by two tables:
$\underline{e^{i\alpha} = +1}$

| $1/\sqrt{2} =$ | $m_0$ | $m_1$ | $m_2$ | $m_3$ | $n_0$ | $n_1$ | $n_2$ | $n_3$ |
|---|---|---|---|---|---|---|---|---|
| $\gamma^0_M = \gamma^0 \gamma^2 \times$ | +1 | +1 | −1 | −1 | +1 | +1 | −1 | −1 |
| $\gamma^1_M = \gamma^1 \gamma^2 \times$ | +1 | +1 | +1 | +1 | −1 | −1 | −1 | −1 |
| $\gamma^2_M = \gamma^2 \times$ | +1 | −1 | −1 | +1 | −1 | +1 | +1 | −1 |
| $\gamma^3_M = \gamma^3 \gamma^2 \times$ | +1 | +1 | +1 | +1 | +1 | −1 | +1 | +1 |

(22)

$\underline{e^{i\alpha} = -1}$

$$
\begin{array}{lcccccccc}
1/\sqrt{2} = & m_0 & m_1 & m_2 & m_3 & n_0 & n_1 & n_2 & n_3 \\
\gamma^0_M = \gamma^0 \gamma^2 \times & -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 \\
\gamma^1_M = \gamma^1 \gamma^2 \times & -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\
\gamma^2_M = \gamma^2 \times & +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 \\
\gamma^3_M = \gamma^3 \gamma^2 \times & -1 & -1 & -1 & -1 & -1 & +1 & -1 & -1
\end{array} \qquad (23)
$$

These Majorana bases are similar to each other: $\gamma^2_M = \pm \gamma^2$ , the remaining three Dirac matrices are multiplied by $\pm \gamma^2$. All 16 possibilities are presented.

## 5. Electrically neutral fermion with anomalous magnetic moment, real or complex nature of the wave function

Let us briefly consider the fermion with additional electromagnetic characteristic, anomalous magnetic moment. The known Petras formalism provides us with the following generally covariant equation( [see [7,8])

$$
\Big\{ \gamma^\alpha(x) \; [ \; i \; (\partial_\alpha + \Gamma_\alpha(x)) + \frac{e}{\hbar c} \; A_\alpha(x) \; ] \; + \\
+ \; \mu \; \left[ -i \frac{e}{\hbar c} \sigma^{\alpha\beta}(x) F_{\alpha\beta}(x) - \frac{1}{4} R(x) \right] \; - \; \frac{mc}{\hbar} \Big\} \; \Psi(x) = 0 \; . \qquad (24)
$$

In the context of the above analysis, special interest may represent one limiting case [10] : namely, when electrical charge $e$ and a free parameter $\mu$ approach zero and infinity respectively, so that the quantity $e\mu$ remains finite. In the same way let $\mu R(x)$ remain finite too:

$$
\underline{\epsilon \Longrightarrow 0}, \qquad e = \epsilon, \qquad \mu = \frac{g}{\epsilon} = \infty \; , \qquad R(x) = 4\epsilon \; r(x) = 0 \; . \qquad (25)
$$

Then, eq. (24) will take the forme

$$
\Big\{ \; i\gamma^\alpha(x) \; (\partial_\alpha + \Gamma_\alpha(x)) + g \; \left[ -\frac{i}{\hbar c} \sigma^{\alpha\beta}(x) F_{\alpha\beta}(x) - r(x) \right] \; - \; \frac{mc}{\hbar} \Big\} \; \Psi(x) = 0 \; . \qquad (26)
$$

which does contain the standard electromagnetic term because the particle is electrically neutral, $e = 0$. Interaction with external electromagnetic field is realized only through magnetic moment term.

## 6. Conclusions

In Majorana basis, the gravitational part of the wave operator

$$
[ \; i\gamma^\alpha(x) \; ( \; \partial_\alpha \; + \; \Gamma_\alpha(x) \; ) \; - g \; r(x) \; ]
$$

is real and it does not mix real and imaginary constituents of the wave function $\Psi_M = \varphi_+ + i\,\varphi_-$ . However, the mixing is done by the magnetic moment term:

$$\Big[\, i\gamma^{\alpha}(x)\ (\ \partial_{\alpha}\ +\ \Gamma_{\alpha}(x)\ )\ -g\, r(x) - \ \frac{mc}{\hbar}\,\Big]\,\varphi_{+} - \frac{1}{\hbar c}\,\sigma^{\alpha\beta}(x)F_{\alpha\beta}(x)\ \varphi_{-} = 0\ ,$$
$$\Big[\, i\gamma^{\alpha}(x)\ (\ \partial_{\alpha}\ +\ \Gamma_{\alpha}(x)\ )\ -g\, r(x) - \ \frac{mc}{\hbar}\,\Big]\,\varphi_{-} + \frac{1}{\hbar c}\,\sigma^{\alpha\beta}(x)F_{\alpha\beta}(x)\ \varphi_{+} = 0\ . \qquad (27)$$

Therefore, if electrically neutral fermion has non-vanishing magnetic moment, its 4-spinor wave function must be complex-valued. It seems that real wave functions of Majorana type can be associated only with a particle without any internal electromagnetic structure. Electrical neutrality is insufficient for the particle be of the Majorana type.

It should be noted that the same peculiarities are seen in the theory of boson particles, of spin 0 and spin 1, with additional electromagnetic structure – polarizability. The wave functions for neutral particles of $S = 0, 1$ with non-vanishing polarizability are complex valued.

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[9] All the matrices in the right-hand side are referred to spinor representation.

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