Nonlinear Dynamics and Applications. Vol. 13 (2006) 140 - 145

The metric energy-momentum tensor for particles with polarizabilities in the electromagnetic field.

S.A. Lukashevich* and N.V. Maksimenko†
Gomel State University
Sovetskay street, 102,
Gomel, 246699, Belarus

Within the covariant lagrangian formalism and relativistic theory of continuous media the metric energy-momentum tensor for spinor particles with polarizabilities in the electromagnetic field have been obtained. The equation of motion for spin-1/2 particles in the external electromagnetic field was determined.

PACS numbers: 13.60.F, 11.10.E, 11.80
Keywords: covariant Lagrangian, energy-momentum tensor

1. Introduction

Interaction of the electromagnetic field with a structural particle in the electrodynamic of hadrons is based on the main principles of the relativistic quantum field theory. In the model conceptions where basically the diagram technique is used a number of features for interaction of photons with hadrons have been determined [1, 2]. However, the diagram technique is mainly employed for description of the electromagnetic processes on a simplest quark systems. In the case of interaction of the electromagnetic field with complex quark-gluon systems in the low-energy region perturbative QCD methods are nonapplicable. That is why the low-energy theorems and sum rules are widely used lately [3–5].

In the present time the low-energy electromagnetic characteristics which connect with hadron structure, such as formfactor and polarizabilities, it is possible to obtain from nonrelativistic theory [4, 5]. Passing on from the nonrelativistic electrodynamic to the relativistic field theory one can make use the correspondence principle. But it is necessary step by step to investigate a transition from the covariant Lagrangian formalism to the Hamiltonian one [6, 7].

Using the covariant Lagrangian of interaction of the electromagnetic field with structural polarizable particle , the equation of motion and energy-momentum tensor have been obtained. The theoretical-field properties of the energy-momentum tensor and the Hamiltonian in the static limit have been determined.

2. Lagrangian

Let's examine the interaction of the electromagnetic field with moving medium in the covariant four-dimensional formulation. In this case it is possible to take advantage from relativistic

*E-mail: belousova@gsu.unibel.by
†E-mail: maksimenko@gsu.unibel.by

Lagrangian of the electrodynamic of continuous media [8]

$$L' = -\frac{1}{2}(e^2 - b^2) - \frac{1}{2}e(\widehat{\varepsilon} - 1)e + \frac{1}{2}b(\widehat{\eta} - 1)b, \tag{1}$$

where $\eta_{\sigma\nu} = (\widehat{\mu}^{-1})_{\sigma\nu}$.

If we take a vector

$$b^\sigma = \mu^{\sigma\nu} h_\nu,$$

then the lagrangian (1) becomes

$$L'' = -\frac{1}{2}(e^2 - h^2) - \frac{1}{2}e(\widehat{\varepsilon} - 1)e + \frac{1}{2}h(\widehat{\mu} - 1)h. \tag{2}$$

Tensors $\varepsilon_{\mu\nu}$ and $\mu_{\mu\nu}$ in expressions (1) and (2) are tensors of the dielectric and magnetic polarizabilities of media in the state of rest. Four-dimensional vectors e^μ and h^μ have components [8, 9]

$$e^\mu \left\{\gamma\left(\mathbf{E}\mathbf{v}\right), \gamma\left(\mathbf{E} - [\mathbf{H}\mathbf{v}]\right)\right\}, \tag{3}$$

$$h^\mu \left\{\gamma\left(\mathbf{H}\mathbf{v}\right), \gamma\left(\mathbf{H} + [\mathbf{E}\mathbf{v}]\right)\right\}, \tag{4}$$

where $\gamma = \frac{1}{\sqrt{1-\mathbf{v}^2}}$, $\mathbf{v}$- moving media velocity; e^μ, h^μ are connected with the electromagnetic field tensors as $e^\mu = F^{\mu\nu}U_\nu$, $h^\mu = \widetilde{F}^{\mu\nu}U_\nu$. Four-dimensional velocity U has components

$$U^\mu \left\{\gamma, \mathbf{v}\gamma\right\}.$$

As the interesting question for us is the interaction of the electromagnetic field with polarizable particle so further it will be more convenient to use lagrangian (2).

We can write down the electromagnetic field tensor through vectors $\mathbf{E}$ and $\mathbf{H}$ [9]

$$\widehat{F} = \begin{pmatrix} 0 & -\mathbf{E} \\ \mathbf{E} & \mathbf{H}^\times \end{pmatrix}, \tag{5}$$

where $(\mathbf{H}^\times)_{ij} = \varepsilon_{ikj}H_k$, $\mathbf{E}$ and $\mathbf{H}$ - are intensity vectors of electric and magnetic fields.

The tensors of electric $\widehat{\alpha}$ and magnetic $\widehat{\beta}$ polarizabilities we inserting via relations:

$$\widehat{\varepsilon} = I + 4\pi\widehat{\alpha}, \quad \widehat{\mu} = I + 4\pi\widehat{\beta}. \tag{6}$$

Thus the lagrangian (2) it is possible to present as [10, 11]

$$L = L_0 + L_I, \tag{7}$$

where $L_0 = -\frac{1}{2}(e^2 - h^2) = -\frac{1}{4}F^2$,

$$L_I = -2\pi(e\widehat{\alpha}e - h\widehat{\beta}h). \tag{8}$$

In this expressions are putting names of variables in the view: $F^2 = F_{\mu\nu}F^{\mu\nu}$, $e\widehat{\alpha}e = e_\mu\alpha^{\mu\nu}e_\nu$, $h\widehat{\beta}h = h_\mu\beta^{\mu\nu}h_\nu$.

The tensors $\alpha^{\mu\nu}$ and $\beta^{\mu\nu}$ it is possible to represent in the diagonal form using metric tensor $g^{\mu\nu}$

$$\alpha^{\mu\nu} = g^{\mu\nu}\alpha, \quad \beta^{\mu\nu} = g^{\mu\nu}\beta.$$

In such a case the lagrangian (8) of interaction of the electromagnetic field with polarizable moving media becomes

$$L_I = -2\pi(\alpha e^2 - \beta h^2). \tag{9}$$

Considering the correlation

$$2(e^2 - h^2) = F^2, \tag{10}$$

we receive

$$L_I = -2\pi \left[(\alpha - \beta)\ e^2 + \frac{\beta}{2} F^2 \right]. \tag{11}$$

If we use the lagrangian (7), expressions (9), (10) and Lagrange's equation

$$\partial_\mu \frac{\partial L}{\partial(\partial_\mu A_\nu)} - \frac{\partial L}{\partial A_\nu} = 0, \tag{12}$$

we find

$$\partial_\mu F^{\mu\nu} = j^{(M)\nu} = -\partial_\mu G_I^{\mu\nu}, \tag{13}$$

where

$$G_I^{\mu\nu} = 4\pi \left[(\alpha - \beta)(e^\mu U^\nu - e^\nu U^\mu) + \beta F^{\mu\nu} \right]. \tag{14}$$

From the equations (13) for media in rest the definitions of charge density and current density for bound charges are follow

$$\rho^{(M)} = -4\pi\alpha(\nabla \mathbf{E}) = -4\pi\alpha \cdot div\mathbf{E}, \tag{15}$$

$$\mathbf{j}^{(M)} = 4\pi(\alpha \cdot \partial_t \mathbf{E} - \beta \cdot rot\mathbf{H}), \tag{16}$$

and the Maxwell's equations for media have the form:

$$\begin{aligned} rot\mathbf{E} = -\partial_t \mathbf{H}, \quad div\mathbf{H} = 0, \\ rot\mathbf{B} = \frac{\partial \mathbf{D}}{\partial t}, \quad div\mathbf{D} = 0, \end{aligned} \tag{17}$$

where **D** and **B** are vectors of electric and magnetic induction accordingly, $\mathbf{D} = \widehat{\varepsilon}\mathbf{E}$, $\mathbf{B} = \widehat{\mu}\mathbf{H}$.

The equations (15) and (16) for media in rest one may note down in the covariant form if we insert the tensor [9]

$$\widehat{M} = \begin{pmatrix} 0 & -\mathbf{P} \\ \mathbf{P} & \mathbf{M}^\times \end{pmatrix}, \tag{18}$$

the components of the tensor (18) are vectors of electric and magnetic polarization **M** **P**. Consequently the charge and current densities in this case become

$$\mathbf{j}^{(M)} = \frac{\partial \mathbf{P}}{\partial t} - rot\mathbf{M}, \quad \rho^{(M)} = -div\mathbf{P}. \tag{19}$$

In such a way the equation of motion for moving media has the form

$$\partial_\mu (F^{\mu\nu} + G_I^{\mu\nu}) = j^\nu, \tag{20}$$

and for media in rest the next form

$$\partial_\mu (F^{\mu\nu} + M^{\mu\nu}) = j^\nu, \tag{21}$$

where j^ν - current density of free charges.

To insert the tensor [8, 12, 13]

$$G^{\mu\nu} = d^\mu U^\nu - d^\nu U^\mu + \varepsilon^{\mu\nu\rho\sigma} U_\rho b_\sigma, \tag{22}$$

where $d^\mu = \varepsilon^{\mu\sigma} e_\sigma$, $b_\rho = \mu_{\rho\sigma} h^\sigma$, the lagrangian (2) for moving media one may to present by way of

$$L = -\frac{1}{4} F^{\mu\nu} G_{\mu\nu} = -\frac{1}{2}(e\widehat{\varepsilon}e - h\widehat{\mu}h). \tag{23}$$

3. The momentum-energy tensor and equation of motion

The lagrangian (2) helps us to determine the canonical momentum-energy tensor. Indeed from the Noether's theorem follows [7]

$$T^{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu A_\rho)}(\partial^\nu A^\rho) - g^{\mu\nu} L. \tag{24}$$

If we substitute (23) into (24) it gives

$$T^{\mu\nu} = -G^{\mu\rho}(\partial^\nu A_\rho) - g^{\mu\nu}\frac{1}{4}(F_{\rho\sigma} G^{\rho\sigma}). \tag{25}$$

Now we determine the metric energy-momentum tensor

$$\widetilde{T}^{\mu\nu} = -G^{\mu\rho}(\partial^\nu A_\rho) - g^{\mu\nu} L + \partial_\rho (A^\nu G^{\mu\rho}). \tag{26}$$

To realize a differentiation in (26) and using the equation of motion (20) we find

$$\widetilde{T}^{\mu\nu} = F^\nu_\rho G^{\mu\rho} + \frac{1}{4} g^{\mu\nu}(F_{\rho\sigma} G^{\rho\sigma}). \tag{27}$$

It follows from the correlation (27) that the energy density for media in rest will be

$$\widetilde{T}^{00} = \omega = \frac{1}{2}(\varepsilon \mathbf{E}^2 + \mu \mathbf{H}^2). \tag{28}$$

Now we pass on to the quantum-mechanical description of interaction of the electromagnetic field with polarizible particles. For this purpose we will use the correspondence principle [6, 7]. Extracting in the lagrangian (23) the contributions of electric and magnetic polarizabilities we note down the lagrangian in the form:

$$L_I = -2\pi(\alpha F_{\mu\rho} F^\mu_\sigma - \beta \widetilde{F}_{\mu\rho} \widetilde{F}^\mu_\sigma) U^\rho U^\sigma, \tag{29}$$

where $\widetilde{F}_{\mu\rho} = \frac{1}{2}\varepsilon_{\mu\rho\sigma\kappa} F^{\sigma\kappa}$, $\varepsilon_{0123} = -1$.

In accordance with the correspondence principle in the equation (29) we realize a replacement

$$U^\rho U^\sigma \to \widetilde{\Theta}^{\rho\sigma} = \frac{1}{2}(\Theta^{\rho\sigma} + \Theta^{\sigma\rho}), \tag{30}$$

where $\Theta^{\rho\sigma}$ is given by

$$\Theta^{\rho\sigma} = \frac{i}{2}\overline{\psi}\gamma^{\rho} \overleftrightarrow{\partial^{\sigma}} \psi, \tag{31}$$

here $\overleftrightarrow{\partial_{\mu}}= \overrightarrow{\partial_{\mu}} - \overleftarrow{\partial_{\mu}}$, γ^{ρ} - the Dirac's matrixes.

As a result for the case of interaction of the electromagnetic field with polarizable structural spin-1/2 particle we find the following lagrangian

$$L_I = \frac{2\pi}{m}\left[\alpha F_{\mu\rho}F^{\mu}_{\sigma} - \beta \widetilde{F}_{\mu\rho}\widetilde{F}^{\mu}_{\sigma}\right]\widetilde{\Theta}^{\rho\sigma}. \tag{32}$$

The explicit form of the lagfangian (32) is agreed with the normalizing of the wave function $\psi(x)$.

Write out now the total lagrangian to define the moving of the charged, polarizable, spinor particle in the electromagnetic field in the form of:

$$L = \frac{i}{2}\overline{\psi}\, \hat{\overleftrightarrow{\partial}}\, \psi - m\overline{\psi}\psi - e\overline{\psi}\widehat{A}\psi - \frac{1}{4}F^2 - \frac{1}{4}F_{\mu\nu}G_I^{(S)\mu\nu}. \tag{33}$$

For representation of the expression (33) was used the correlation

$$\widetilde{F}_{\mu\rho}\widetilde{F}^{\mu\sigma} = F_{\mu\rho}F^{\mu\sigma} - \frac{1}{2}\delta^{\sigma}_{\rho}F_{\mu\nu}F^{\mu\nu}.$$

Then the tensor (14) is determined by

$$G_I^{(S)\mu\nu} = -\frac{4\pi}{m}\left\{(\alpha-\beta)\left[F^{\mu\sigma}\widetilde{\Theta}^{\nu}_{\sigma} - F^{\nu\sigma}\widetilde{\Theta}^{\mu}_{\sigma}\right] + \beta\widetilde{\Theta}^{\rho}_{\rho}F^{\mu\nu}\right\}. \tag{34}$$

Using the lagrangian (33) and antisymmetric tensor (34) the metric momentum-energy tensor we shall define as

$$\widetilde{T}^{\mu\nu} = \widetilde{\Theta}^{\mu\nu} + F^{\nu}_{\rho}F^{\mu\rho} + \frac{1}{4}g^{\mu\nu}F^2 - \frac{e}{2}\overline{\psi}(\gamma^{\mu}A^{\nu} + \gamma^{\nu}A^{\mu})\psi + \widetilde{T}_I^{\mu\nu}, \tag{35}$$

where

$$\widetilde{T}_I^{\mu\nu} = F^{\nu}_{\rho}G_I^{(S)\mu\rho} + \frac{1}{4}g^{\mu\nu}(F_{\rho\sigma}G_I^{(S)\rho\sigma}). \tag{36}$$

It follows from the expression (36) that if the particle impulse is equal to zero then for such particle interacting with the electromagnetic field the interaction energy because of polarizability will be have the form [5]

$$\widetilde{T}_I^{00} = -2\pi(\alpha\mathbf{E}^2 + \beta\mathbf{H}^2). \tag{37}$$

The equation of motion which follows from the lagrangian (33) is given by

$$\partial_{\mu}F^{\mu\nu} = e\overline{\psi}\gamma^{\nu}\psi + j^{(M)\nu}, \tag{38}$$

here $j^{(M)\nu} = -\partial_{\mu}G_I^{(S)\mu\nu}$.

4. Conclusion

The correlations between the covariant Lagrangian and canonical energy-momentum tensor have been determined. This fact on the basis of the correspondence principle has given a proper definition of the low-energy presentation of the Lagrangian function.

To assume the covariant Lagrangian of interaction of the electromagnetic field with a polarizable particles as a basis in the Lagrangian covariant formalism the equation of motion have been found.

References

[1] Brodsky S. J., Primack J.R., Annals of Physics, **Vol.52**, (1969), 315.
[2] Scherer S., Korchin A. Yu., Koch J.H., Phys. Rev., **C54**, (1996), 904.
[3] Levchuk M. I., Moroz L.G., Vesti AN BSSR Ser.: fiz.-mat. nauk 1, (1985), 49.
[4] L'vov A.J., Petrun'kin V.A., Phys. Rev., **C55**, (1997), 359.
[5] Babusci D., Jiordano J., L'vov A.J., Matone J., Nathan A.N., Phys. Rev., **C58**, (1998), 1013.
[6] Veil' G., The theory of groups and quantum mechanics. – Moscow: Nauka, (1986), 495. (in Russian)
[7] A.A.Bogush, L.G.Moroz, Introduction to theory of classic fields. – Nauka i tehnika.– Minsk., (1968), 386. (in Russian)
[8] Bokut' B.V., Serdukov A.N., Jour. of Appl. Spectr., **Vol. 11, N4**, (1969), 704. (in Russian)
[9] F.I.Fedorov, Gyrotropy theory.– Minsk: Nauka i tehnika, (1973), 456. (in Russian)
[10] Maksimenko N.V., Moroz L.G., Proc. 11 Int. School on High Energy Physics and Relativistic Nucl. Phys. (D2-11707)– Dubna., 1979, 533.
[11] Maksimenko N.V, Deruzhkova O.M., Lukashevich S.A., Proc. of Int. School-Seminar "Actual Problems of Particles Physics". – Dubna.,**Vol. II**, 2002, 145.
[12] Luboshic V.L., Smorodinsky Y.A., JETF., **Vol. 42, N3**, (1962) 846. (in Russian)
[13] Preti G., Phys. Rev., **D70**, (2004), 024012.